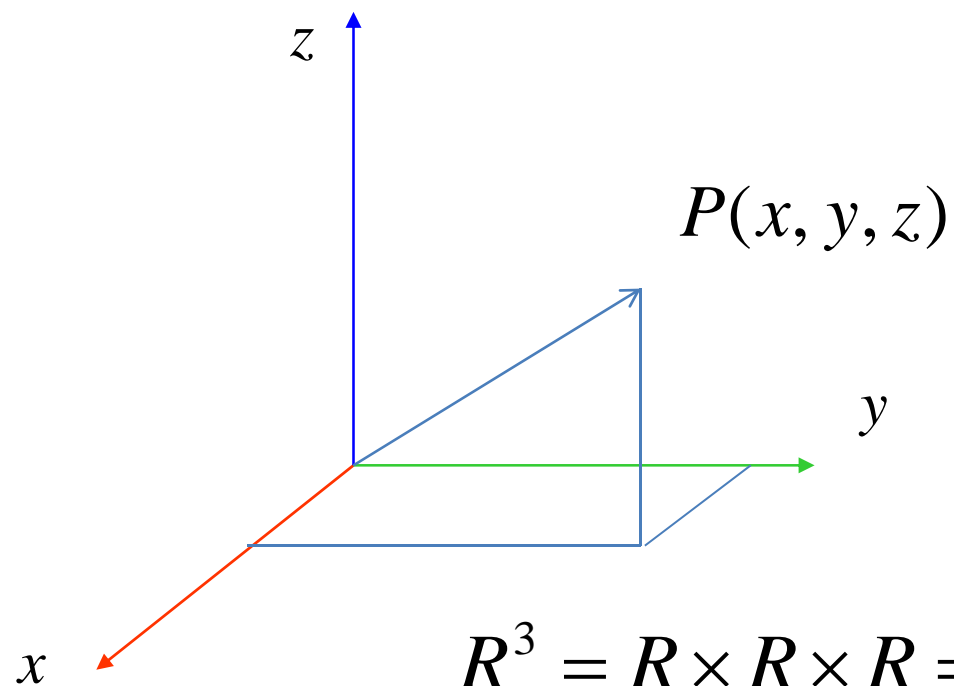


KOORDINATNI SISTEMI U PROSTORU

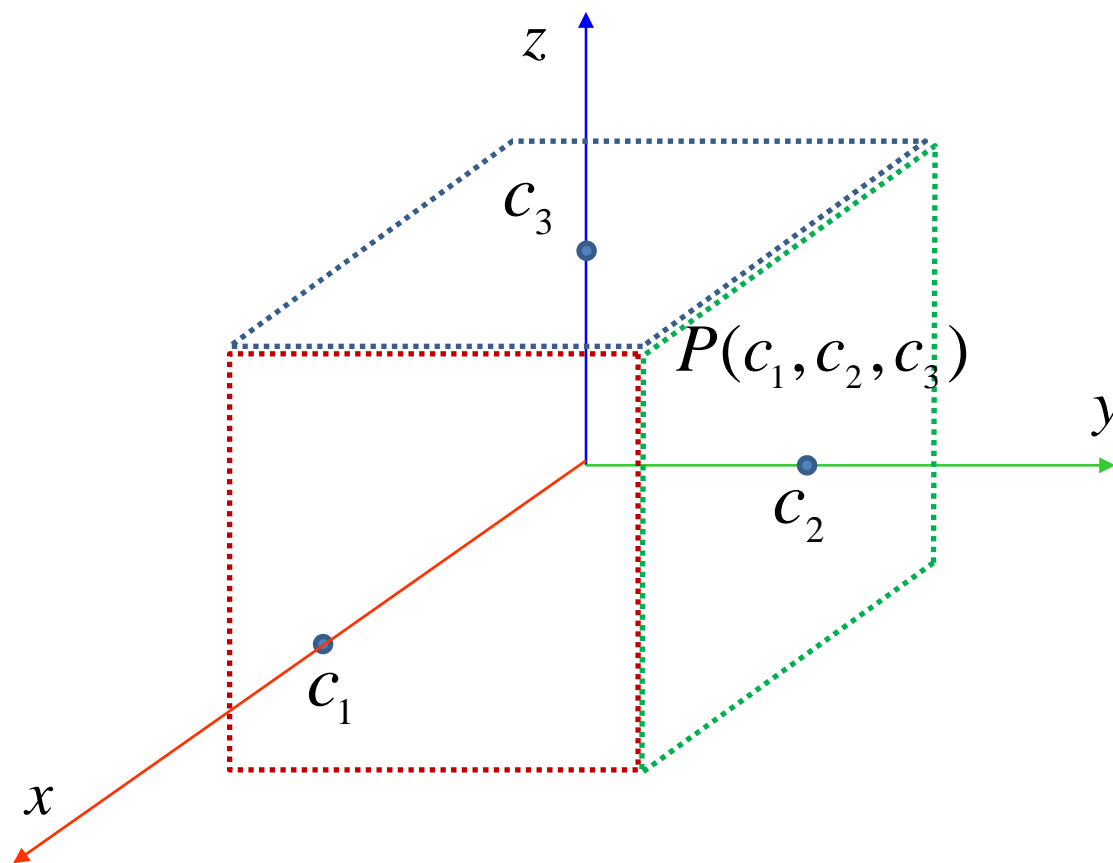
Dekartov pravougli koordinatni sistem



$$R^3 = R \times R \times R = \{(x, y, z) \mid x, y, z \in R\}$$

KOORDINATNI SISTEMI U PROSTORU

Dekartov pravougli koordinatni sistem



$P(x, y, z)$

$$x = \text{const} \quad x = c_1$$

ravan

$$y = \text{const} \quad y = c_2$$

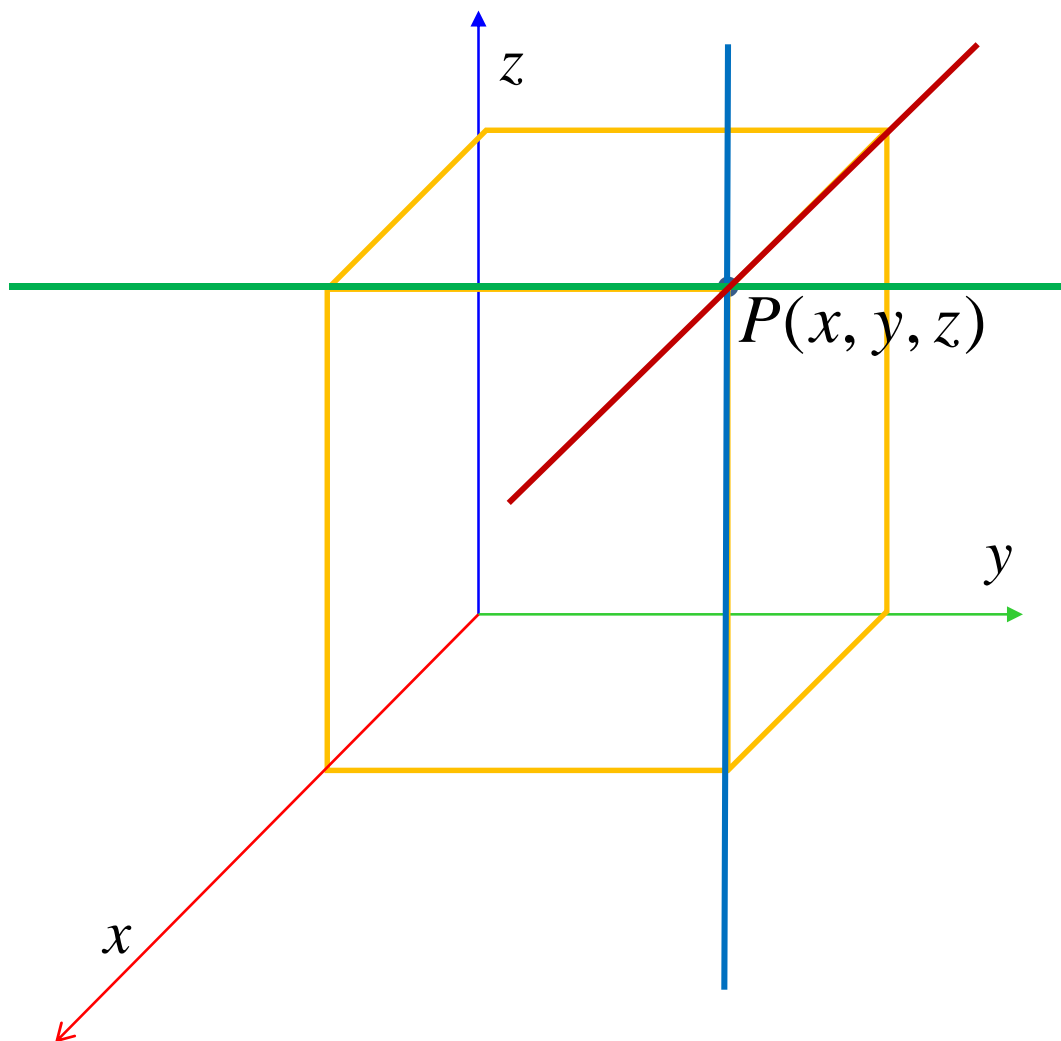
ravan

$$z = \text{const} \quad z = c_3$$

ravan

KOORDINATNI SISTEMI U PROSTORU

Dekartov pravougli koordinatni sistem



$$P(x, y, z)$$

$$x = c_1$$

$$y = c_2$$

prava

$$x = c_1$$

$$z = c_3$$

prava

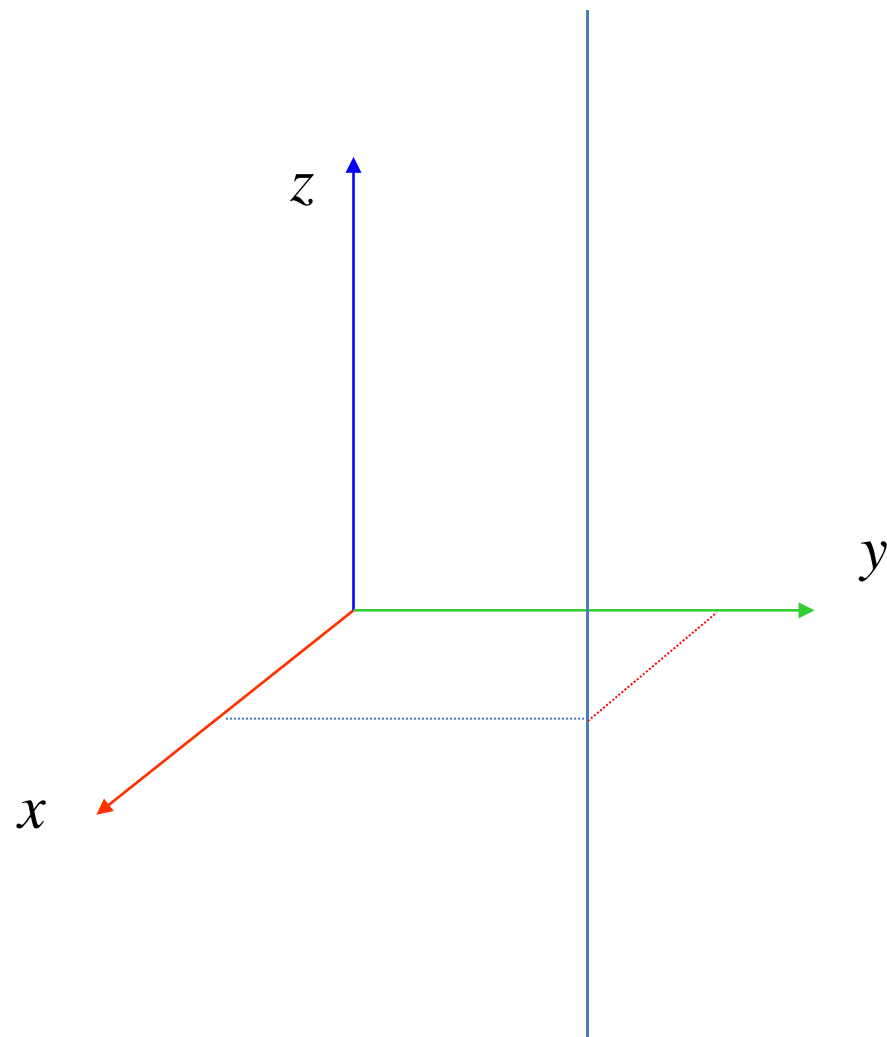
$$y = c_2$$

$$z = c_3$$

prava

KOORDINATNI SISTEMI U PROSTORU

Dekartov pravougli koordinatni sistem



$$\underline{P(x, y, z)}$$

$$x = c_1 \quad y = c_2$$

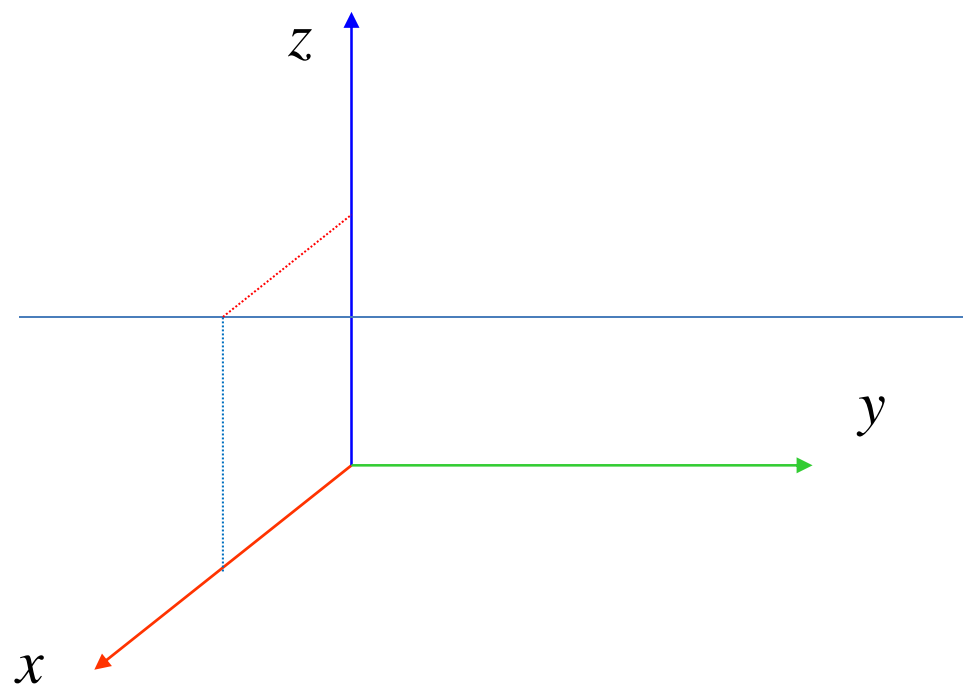
Prava paralelna sa z-osom

$$\underline{P(c_1, c_2, 0)}$$

U Oxy ravni

KOORDINATNI SISTEMI U PROSTORU

Dekartov pravougli koordinatni sistem



$$P(x, y, z)$$

$$x = c_1 \quad z = c_3$$

Prava paralelna sa z-osom

$$P(c_1, 0, c_3)$$

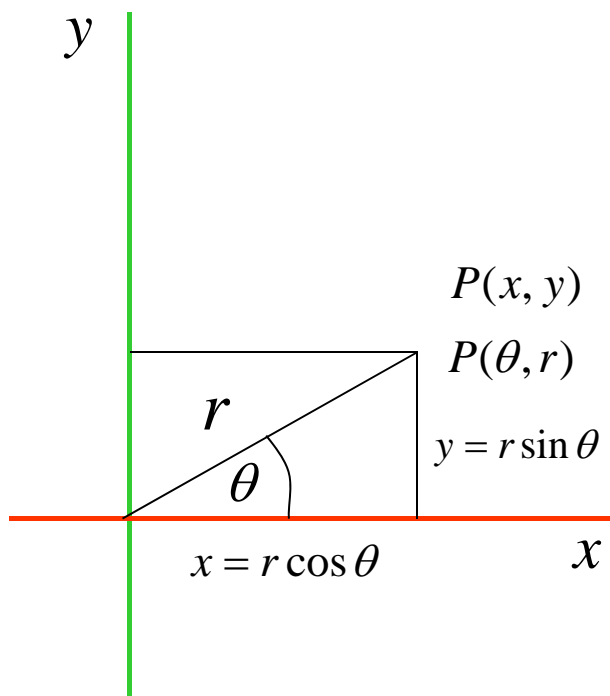
U Oxy ravni

KOORDINATNI SISTEMI U PROSTORU

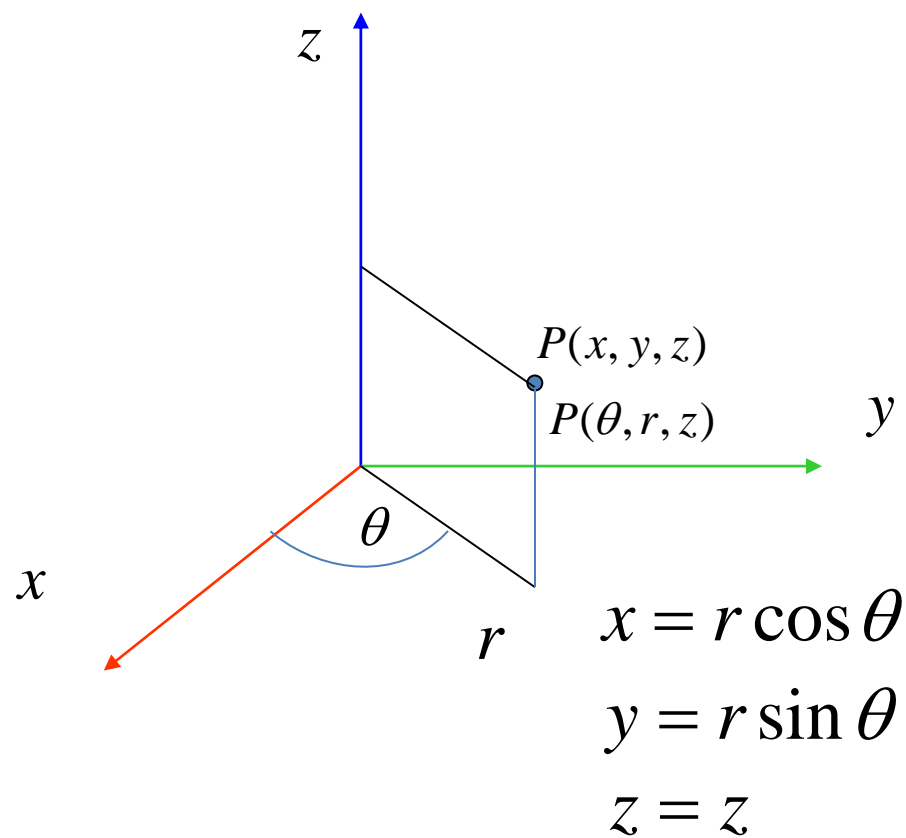
Cilindrični koordinatni sistem

Polarni koordinatni sistem u ravni

U ravni Oxz polarni koordinatni sistem

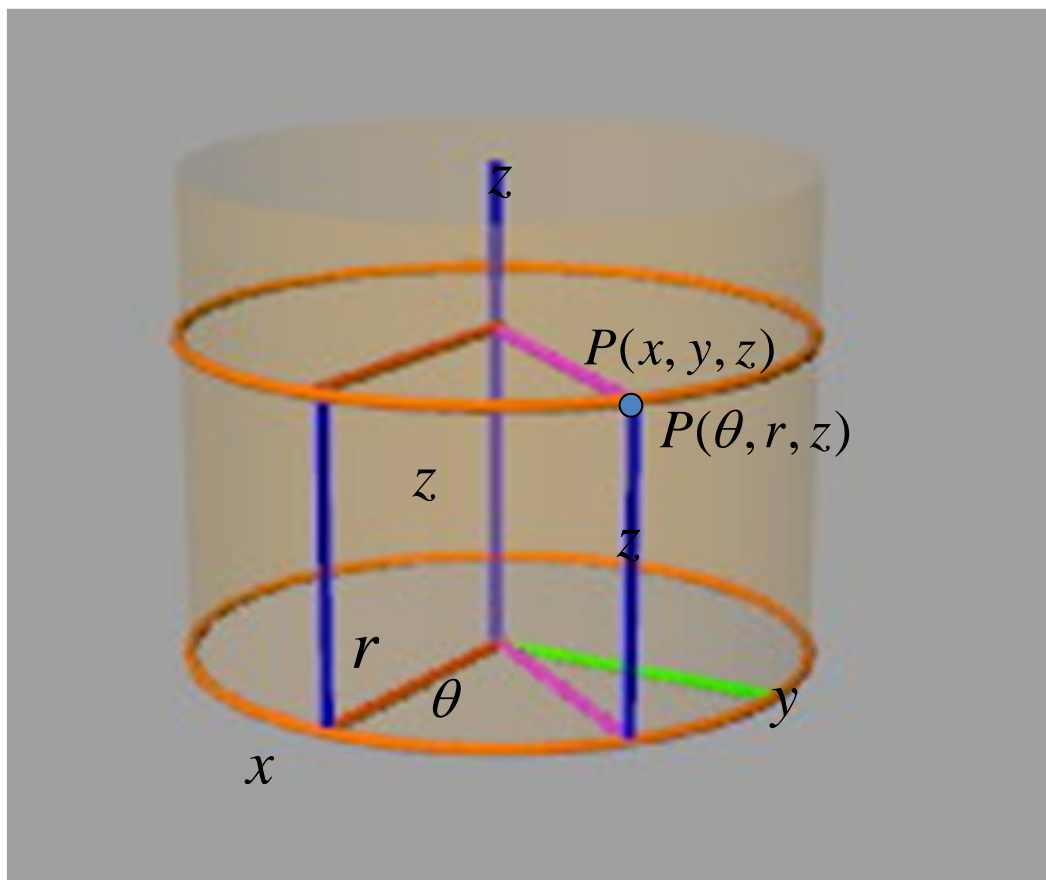


Cilindrični koordinatni sistem



KOORDINATNI SISTEMI U PROSTORU

Cilindrični koordinatni sistem



$$P(x, y, z)$$

$$P(\theta, r, z)$$

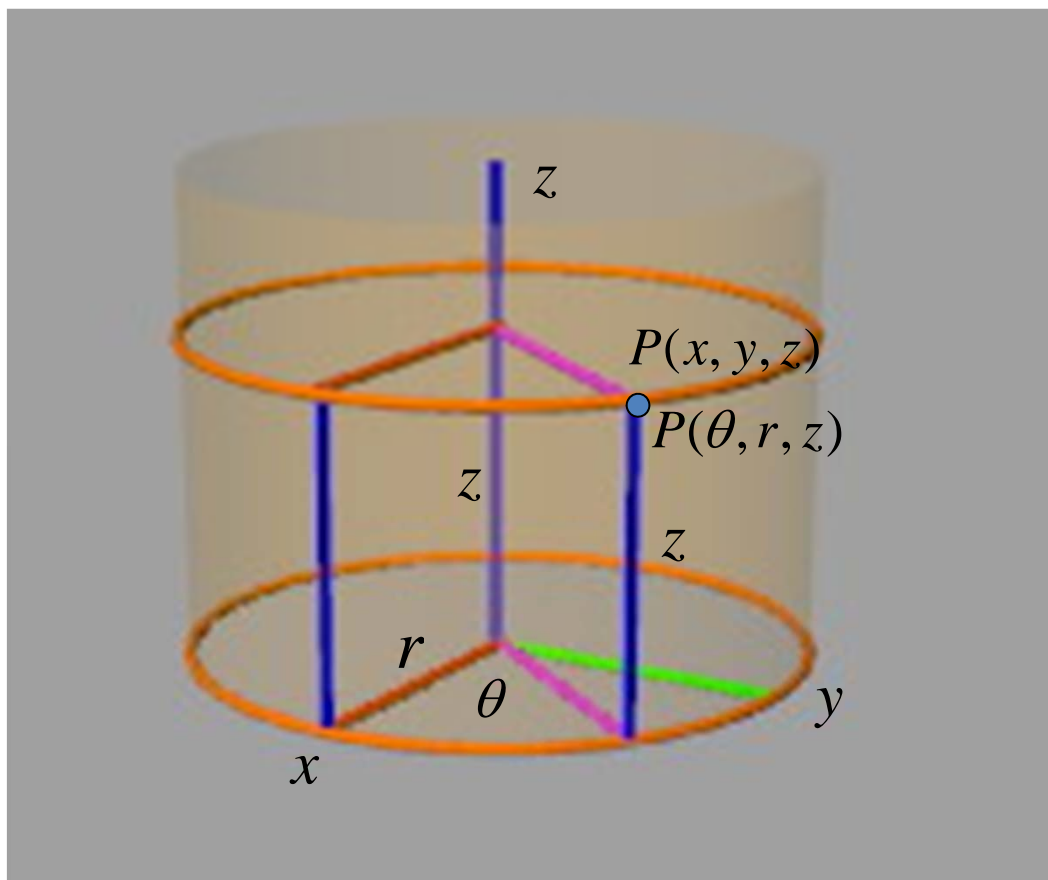
$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

KOORDINATNI SISTEMI U PROSTORU

Cilindrični koordinatni sistem



$$P(x, y, z)$$

$$P(\theta, r, z)$$

$$\theta = \text{const}$$

poluravan

$$r = \text{const}$$

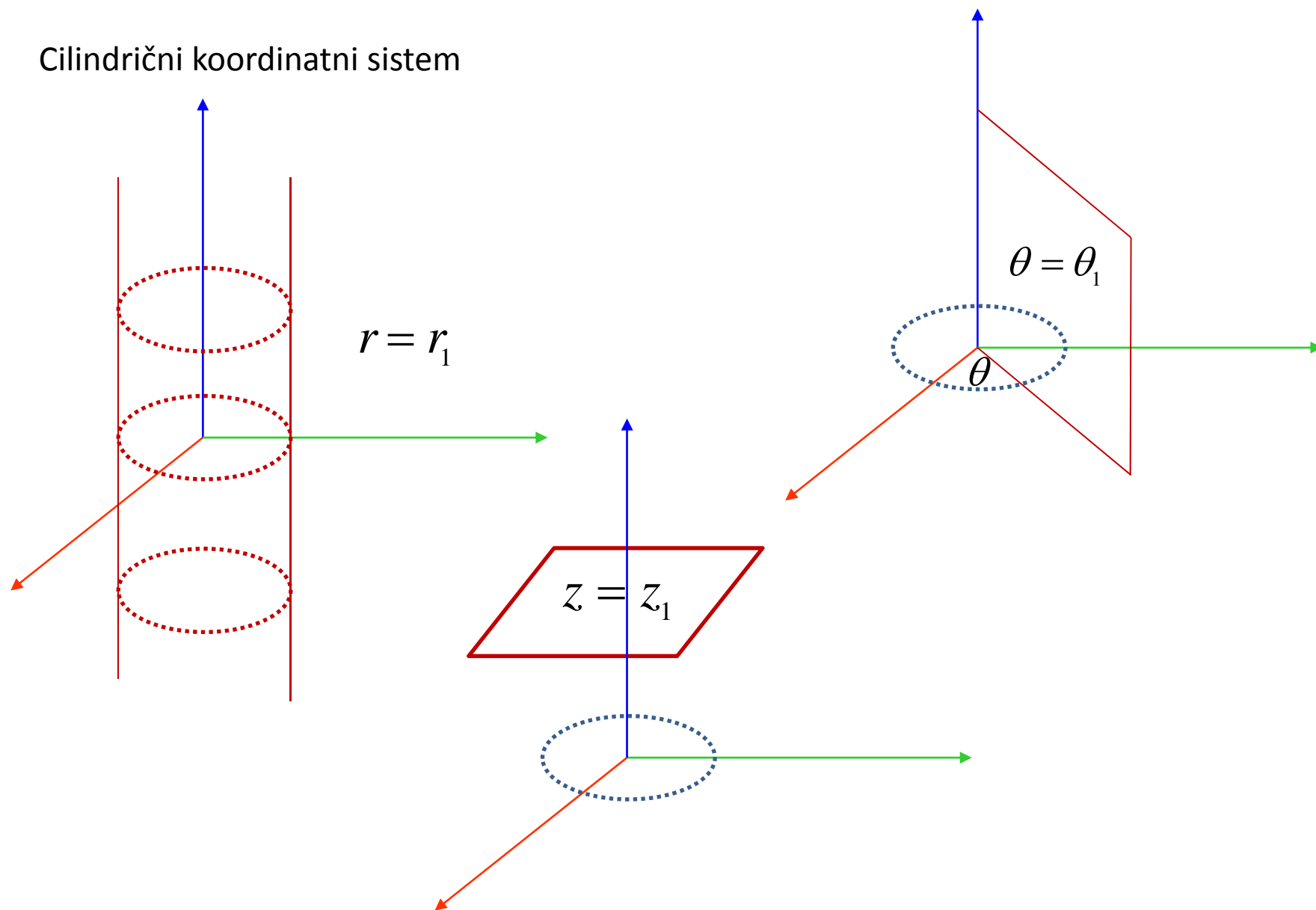
cilindar

$$z = \text{const}$$

ravan

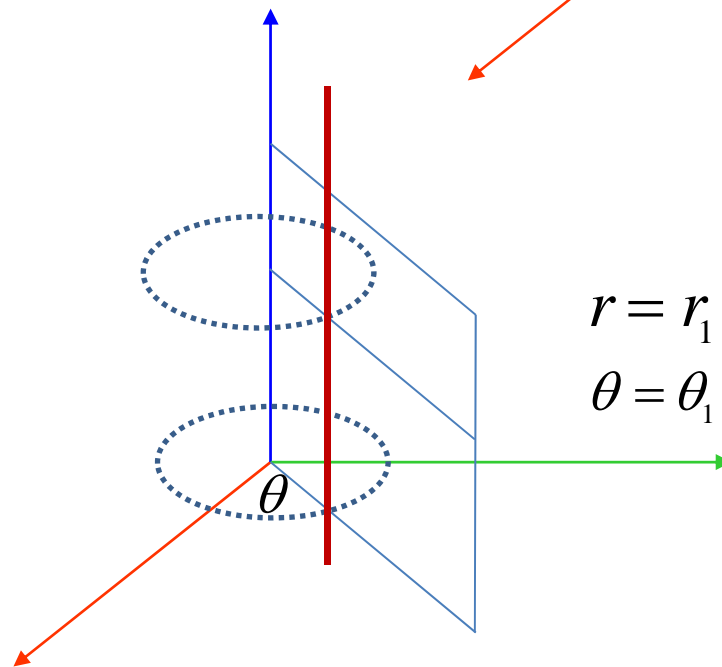
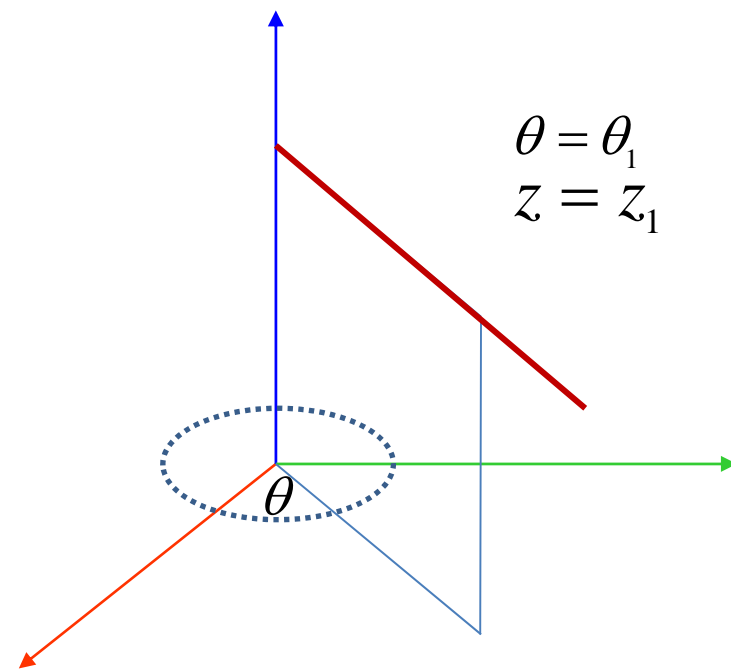
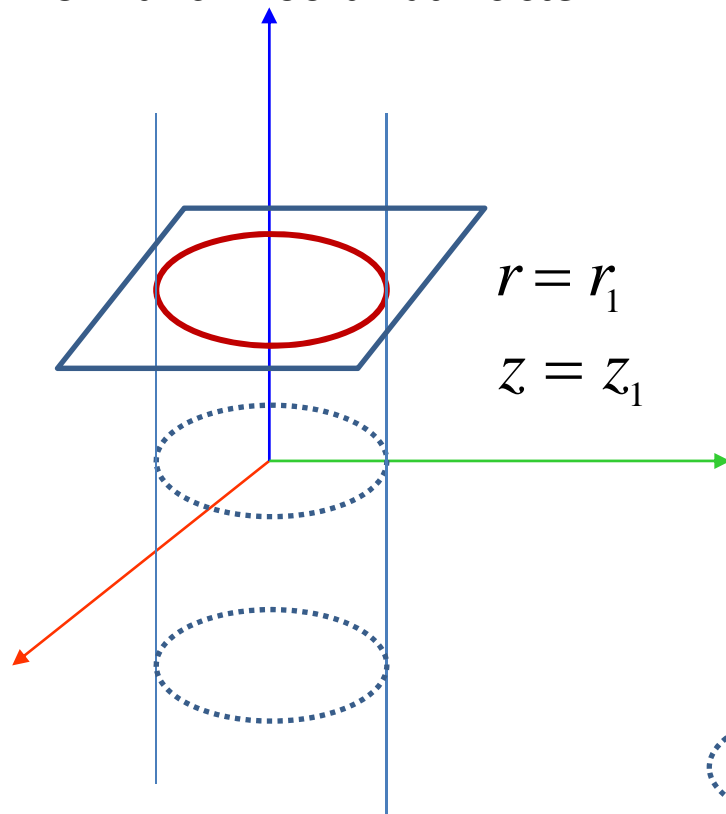
KOORDINATNI SISTEMI U PROSTORU

Cilindrični koordinatni sistem



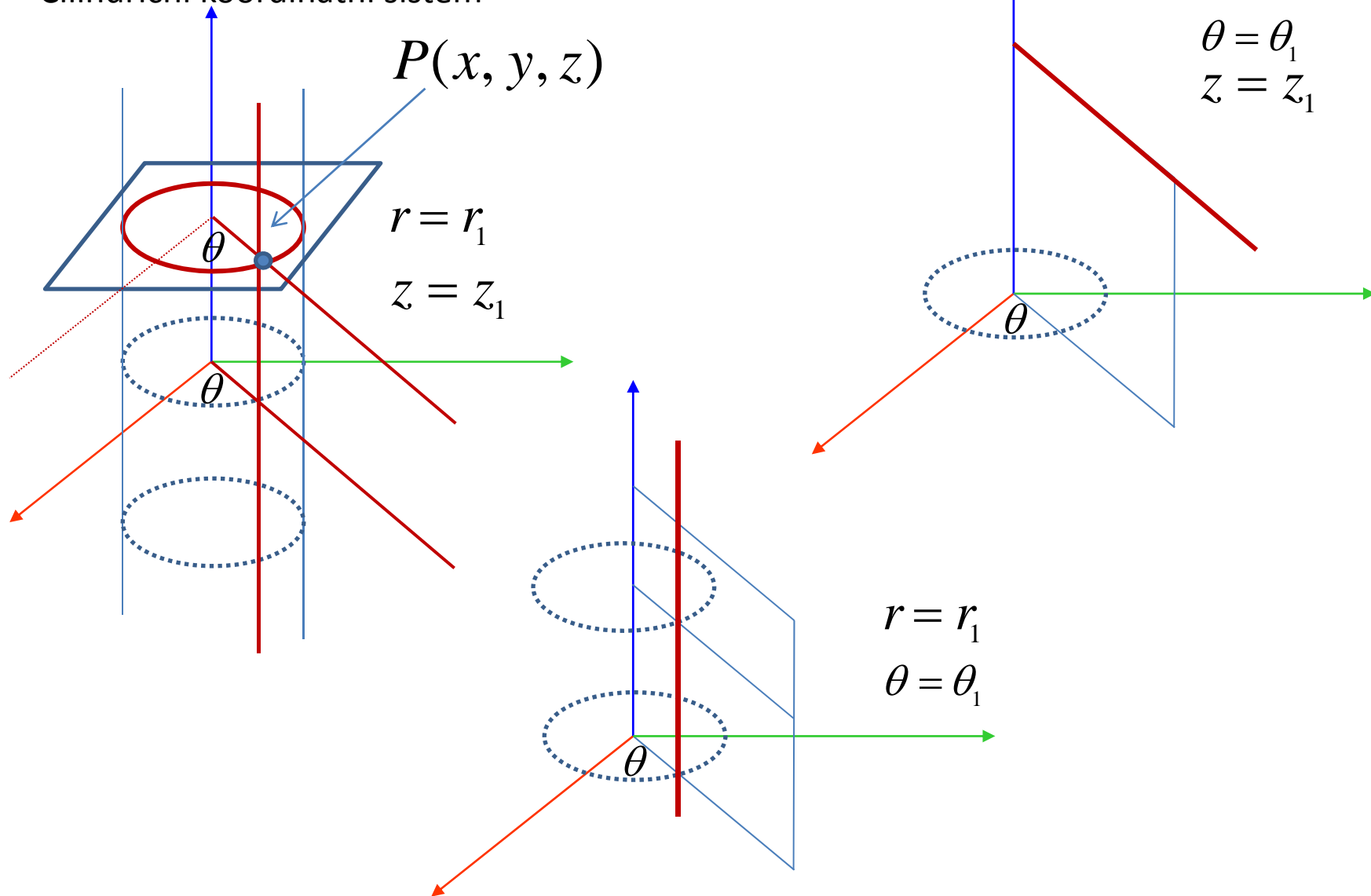
KOORDINATNI SISTEMI U PROSTORU

Cilindrični koordinatni sistem



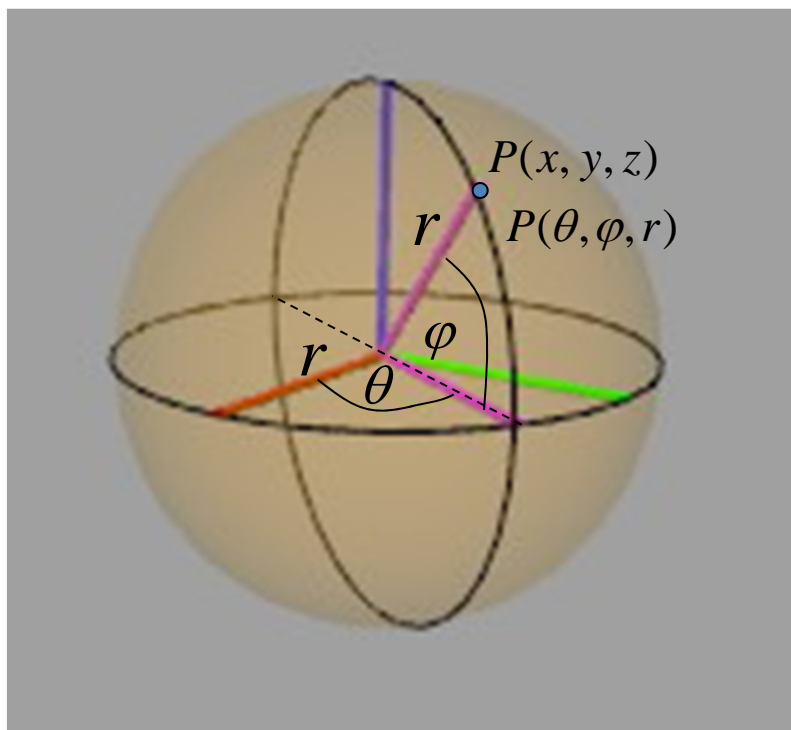
KOORDINATNI SISTEMI U PROSTORU

Cilindrični koordinatni sistem



KOORDINATNI SISTEMI U PROSTORU

Sferni koordinatni sistem



$$P(x, y, z)$$

$$P(\theta, \varphi, r)$$

Odnos Dekartovih i sfernih koordinata

$$x = r \cos \varphi \cos \theta$$

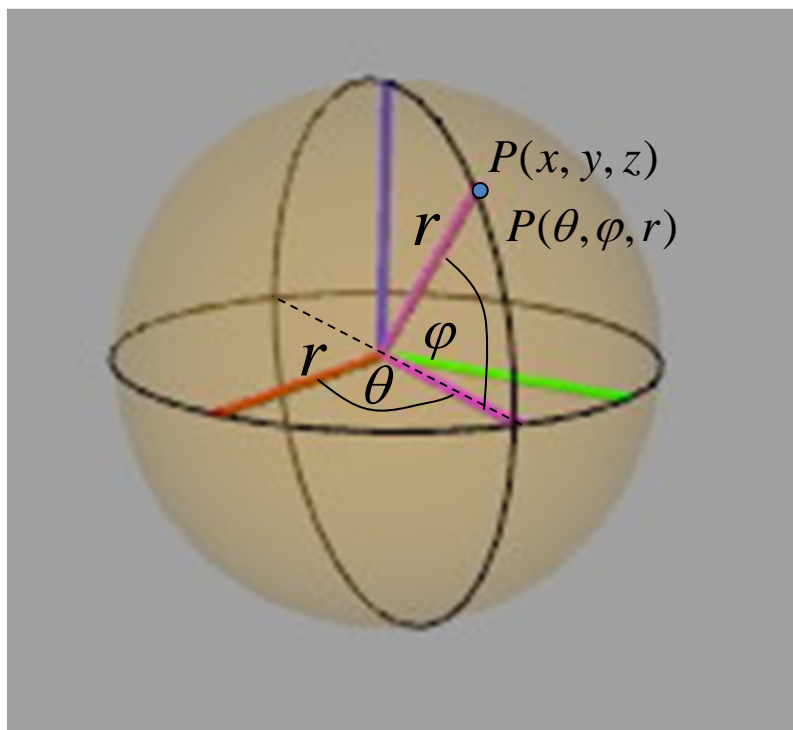
$$y = r \cos \varphi \sin \theta$$

$$z = r \sin \varphi$$

$$0 \leq \theta \leq 2\pi \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

KOORDINATNI SISTEMI U PROSTORU

Sferni koordinatni sistem



$$0 \leq \theta \leq 2\pi \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

$$P(x, y, z)$$

$$P(\theta, \varphi, r)$$

$$\theta = \text{const}$$

poluravan

$$\varphi = \text{const}$$

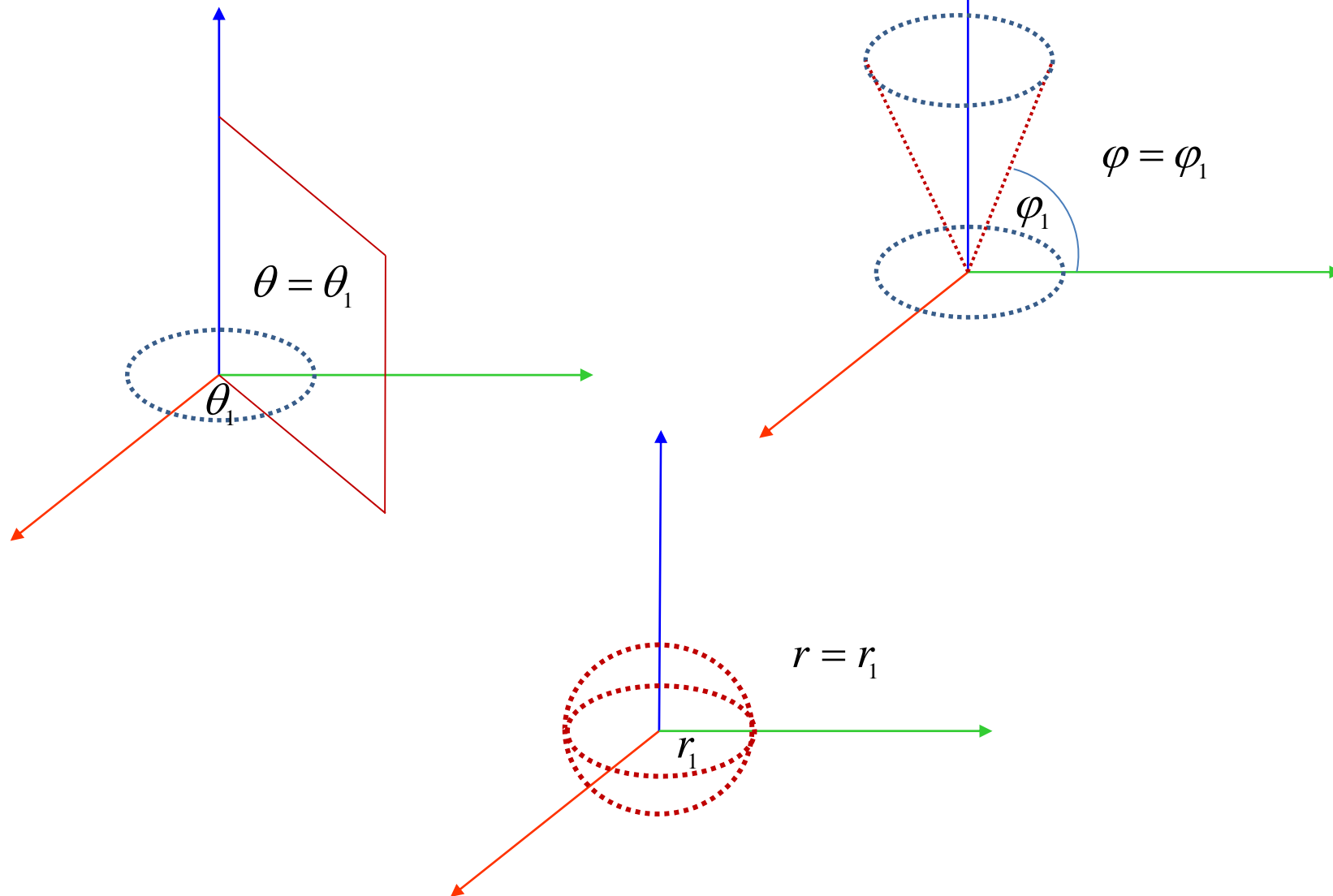
konus

$$r = \text{const}$$

sfera

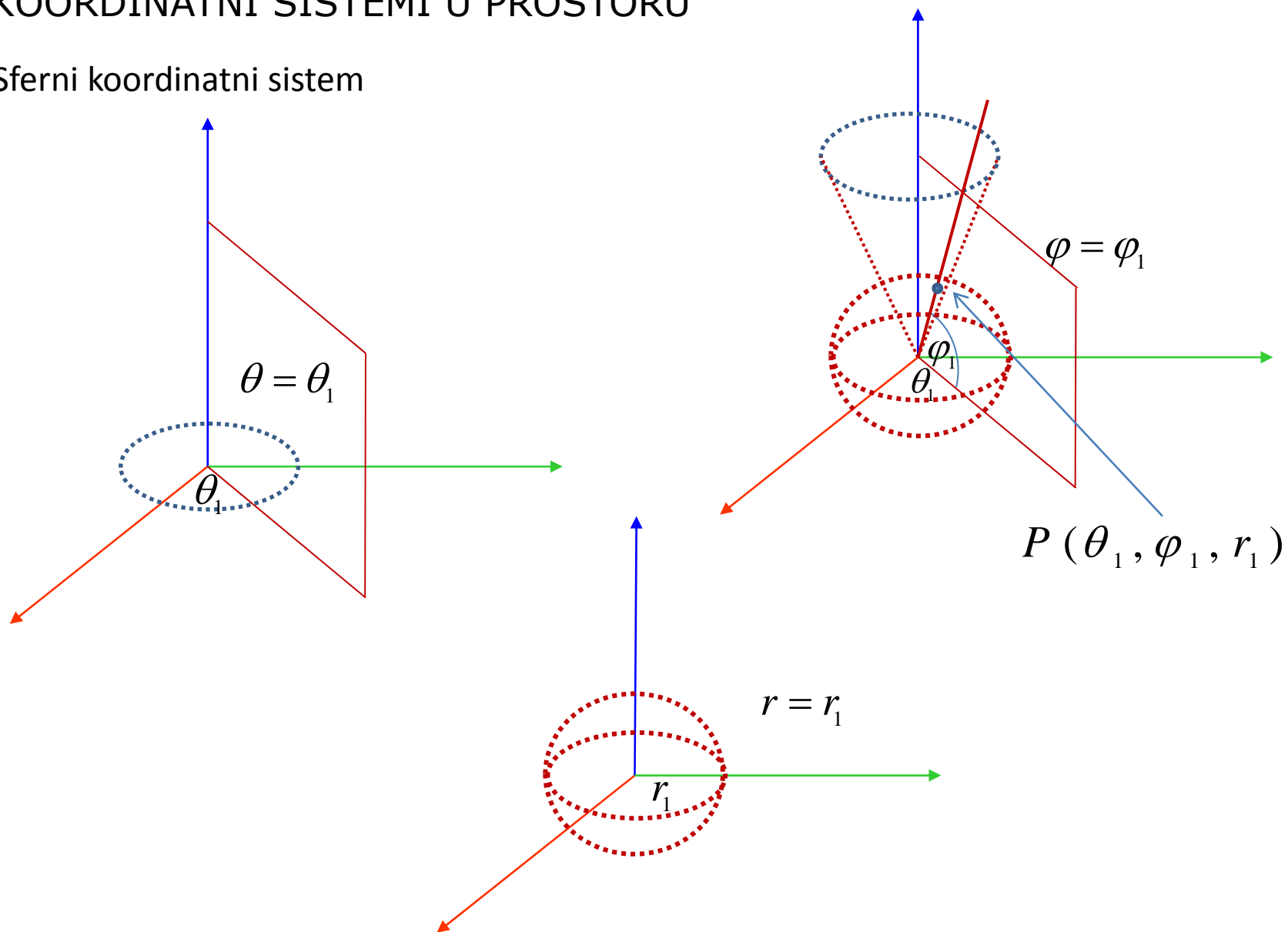
KOORDINATNI SISTEMI U PROSTORU

Sferni koordinatni sistem



KOORDINATNI SISTEMI U PROSTORU

Sferni koordinatni sistem



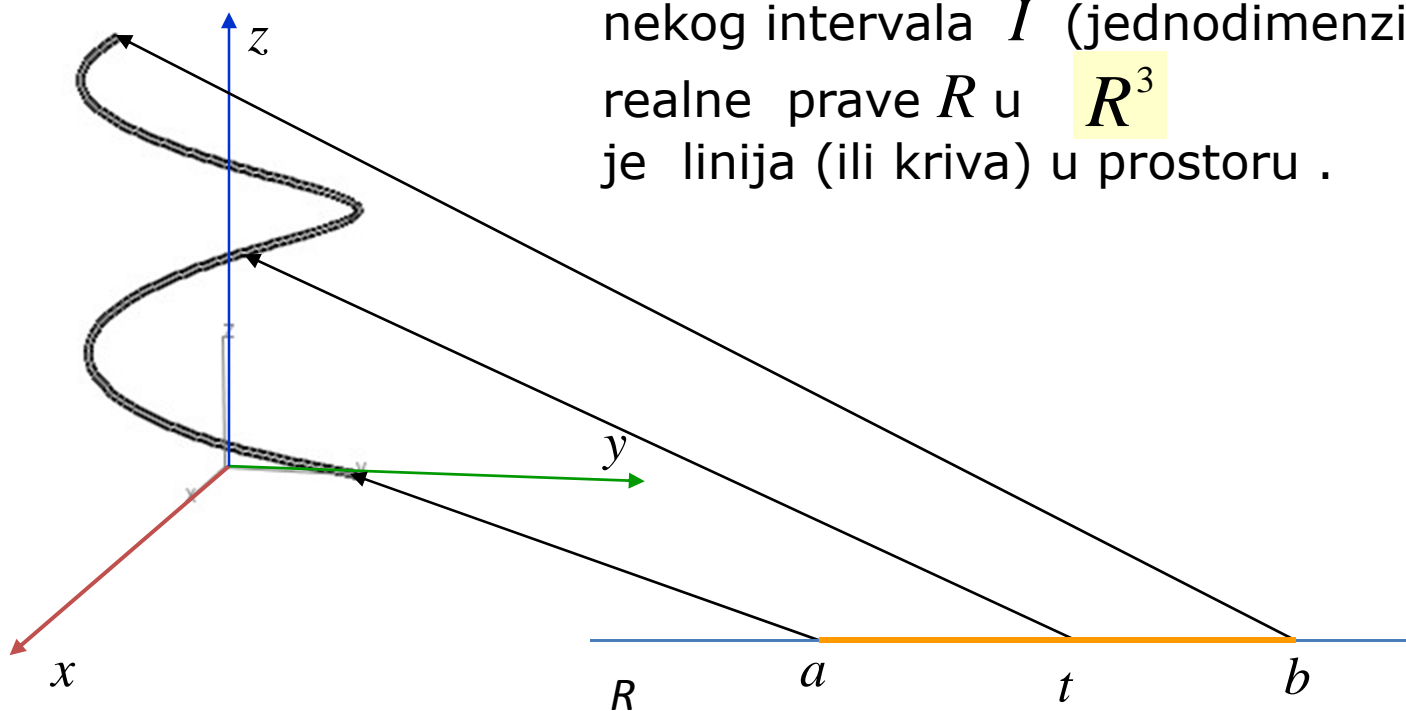
LINIJE (KRIVE) U PROSTORU

Preslikavanje

$$I = [a, b] \subset \mathbb{R}$$

$$\alpha: I \rightarrow \mathbb{R}^3, \quad t \in I$$

nekog intervala I (jednodimenzionalne)
realne prave \mathbb{R} u \mathbb{R}^3
je linija (ili kriva) u prostoru .



\mathbb{R} – parametarska prava

LINIJE (KRIVE) U PROSTORU

$$\alpha : I \rightarrow R^3$$

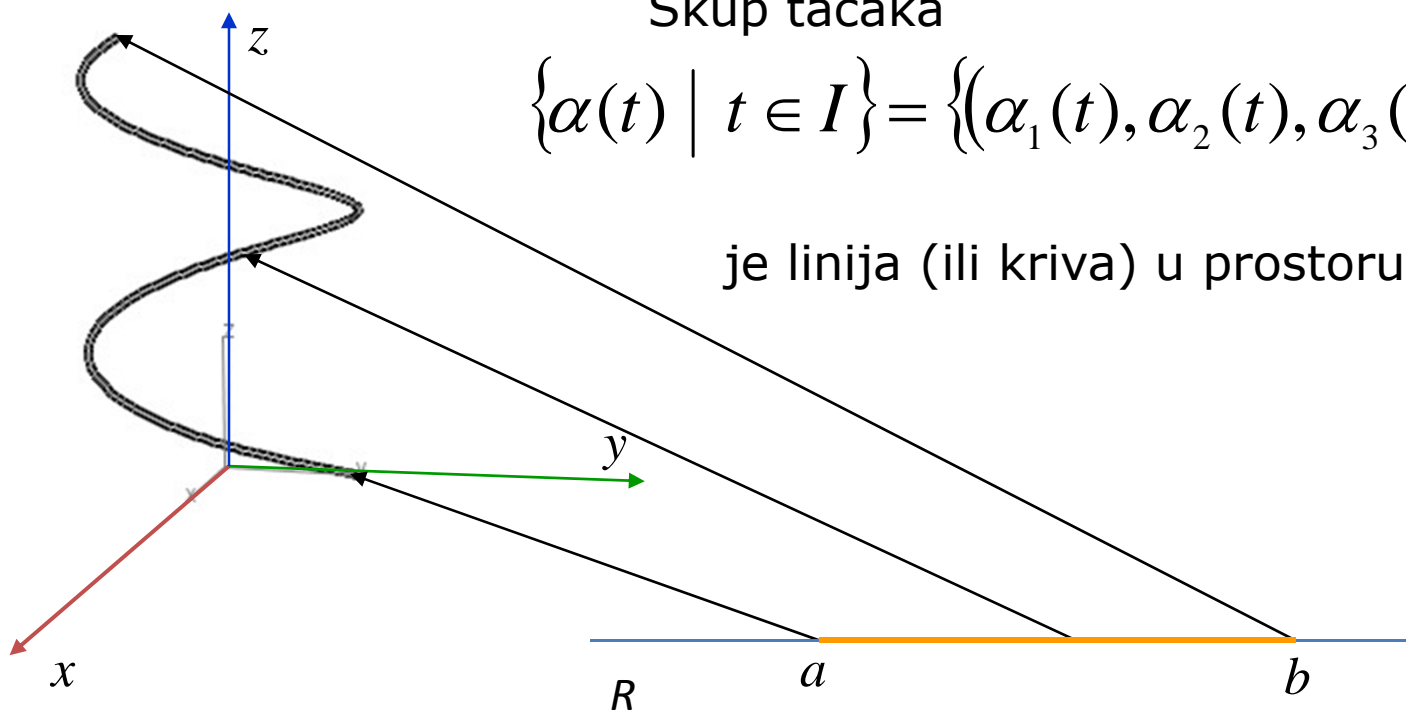
$$I = [a, b] \subset R$$

$$t \in R \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

Skup tačaka

$$\{\alpha(t) \mid t \in I\} = \{(\alpha_1(t), \alpha_2(t), \alpha_3(t)) \mid t \in I\}$$

je linija (ili kriva) u prostoru.



R – parametarska prava

LINIJE (KRIVE) U PROSTORU

$$\alpha : I \rightarrow \mathbb{R}^3$$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

$$a \leq t \leq b$$

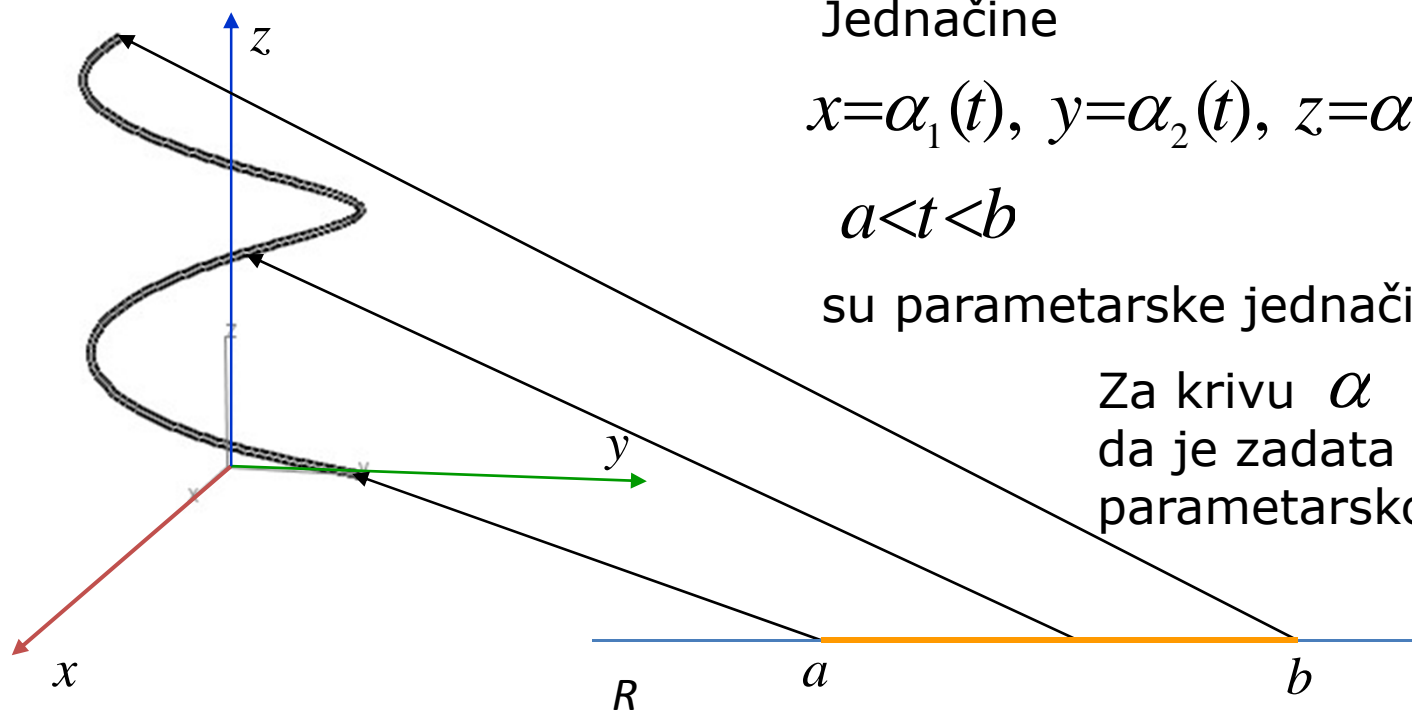
Jednačine

$$x = \alpha_1(t), \quad y = \alpha_2(t), \quad z = \alpha_3(t)$$

$$a < t < b$$

su parametarske jednačine te krive.

Za krivu α se kaže da je zadata u parametarskom obliku.



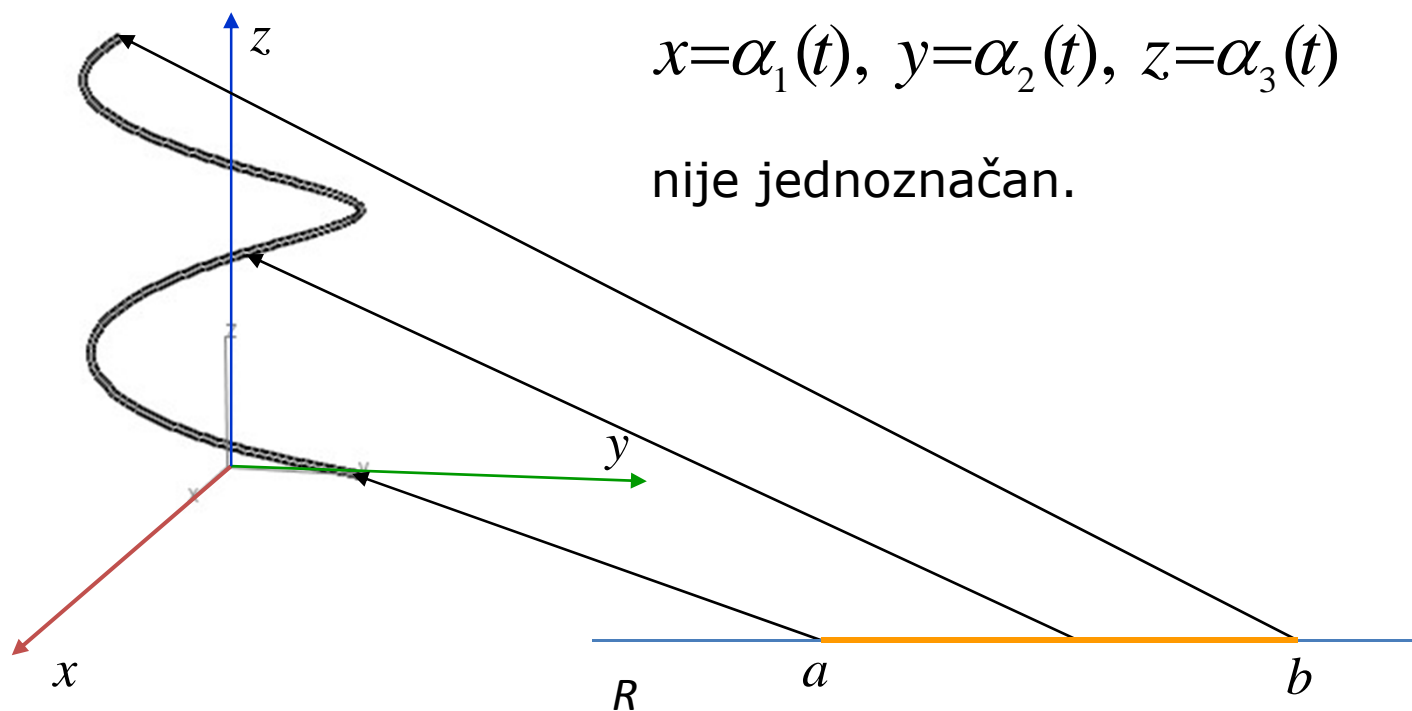
R – parametarska prava

LINIJE (KRIVE) U PROSTORU

Matematički prikaz krive linije C
u prostoru u parametarskom
obliku

$$x=\alpha_1(t), y=\alpha_2(t), z=\alpha_3(t) \quad a < t < b$$

nije jednoznačan.



LINIJE (KRIVE) U PROSTORU

Ukoliko je

$$\varphi : J \rightarrow I$$

Diferencijabilna rastuća funkcija koja vrši preslikavanje nekog intervala $J = [c, d]$ neke parametarske prave τ na interval $I = [a, b]$ tada su

$$\varphi'(\tau) > 0$$

$$x = \alpha_1(\varphi(\tau)), y = \alpha_2(\varphi(\tau)), z = \alpha_3(\varphi(\tau)) \quad c < \tau < d$$

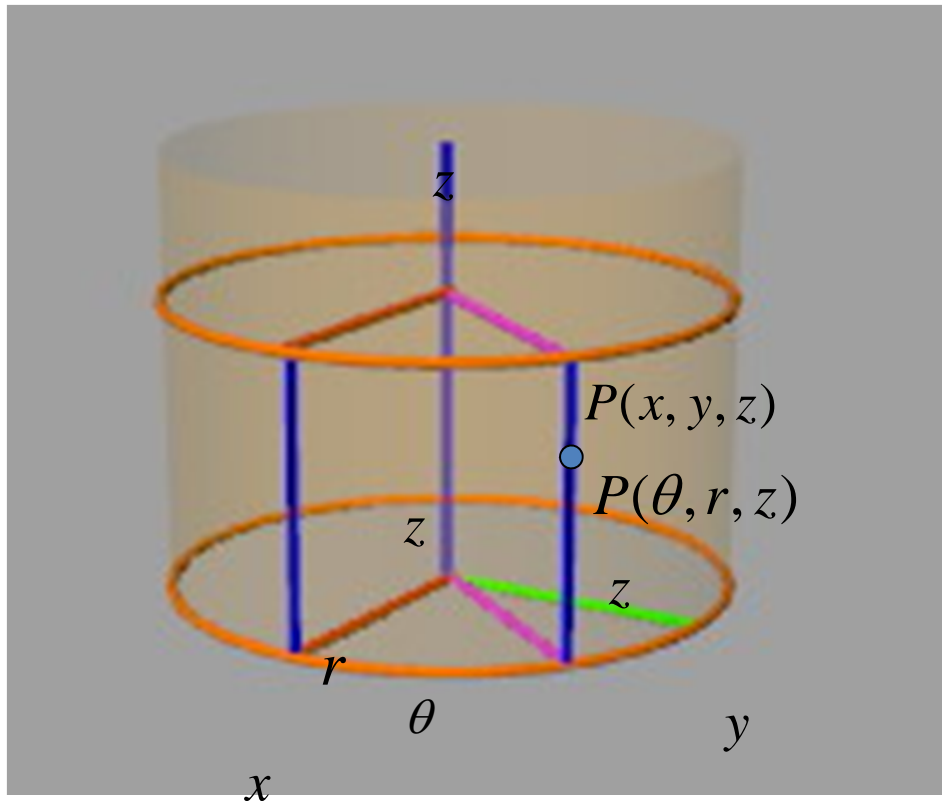
takodje parametarske jednačine iste krive C .

Kaže se da je izvršena reparametrizacija i nove jednačine su:

$$x = \beta_1(\tau), y = \beta_2(\tau), z = \beta_3(\tau) \quad c < \tau < d$$

LINIJE (KRIVE) U PROSTORU

Cilindrični koordinatni sistem



Parametarski zadana kriva
linija u prostoru u
cilindričnom koordinatnom
sistemu:

$$\theta = \varphi_1(t)$$

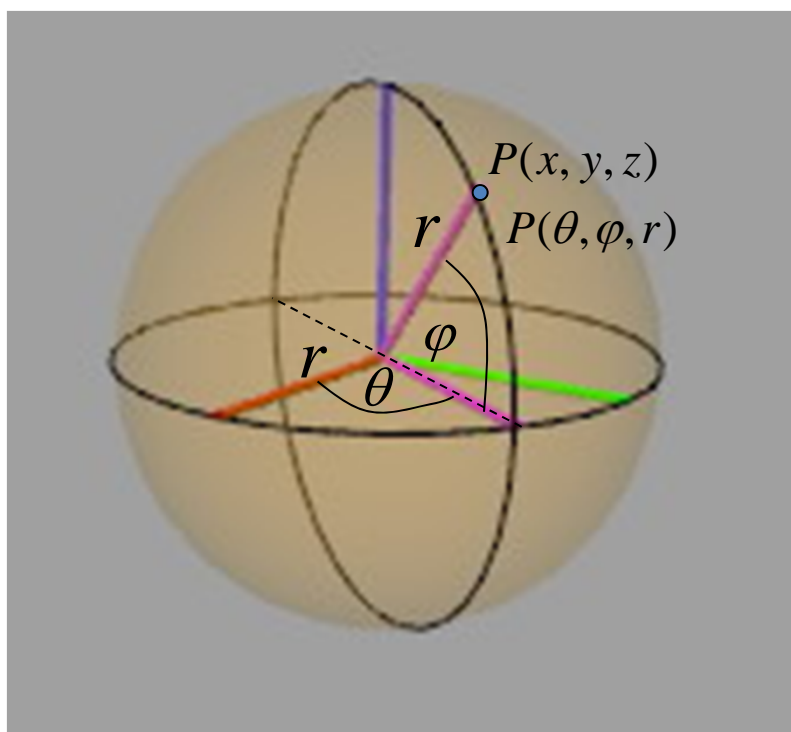
$$r = \varphi_2(t)$$

$$z = \varphi_3(t)$$

$$t_1 \leq t \leq t_2$$

LINIJE (KRIVE) U PROSTORU

Sferni koordinatni sistem



$$0 \leq \theta \leq 2\pi \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Parametarski zadana kriva
linija u prostoru u sfernom
koordinatnom sistemu:

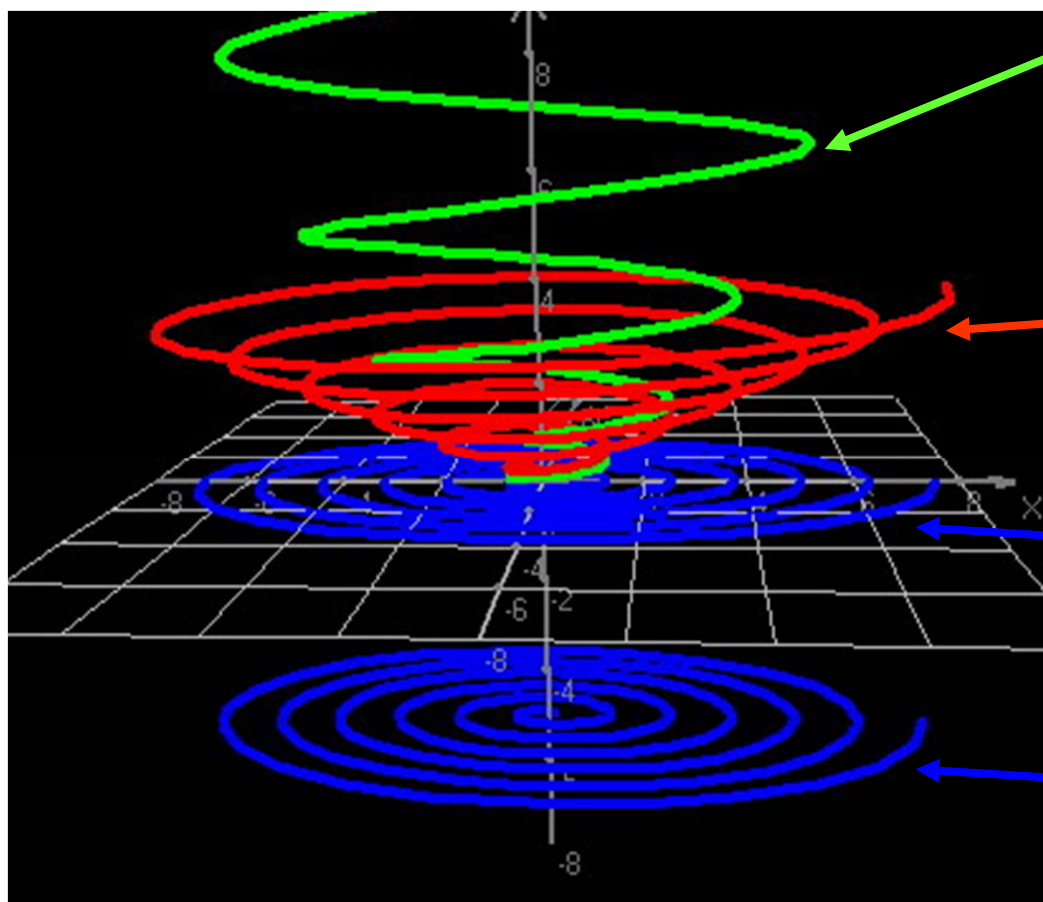
$$\theta = \psi_1(t)$$

$$\varphi = \psi_2(t) \quad t_1 \leq t \leq t_2$$

$$r = \psi_3(t)$$

LINIJE (KRIVE) U PROSTORU

Primeri



4

$$\begin{aligned}x &= 0.2t \cos t \\y &= 0.1t \sin t \\z &= 0.01t^2\end{aligned} \quad 0 \leq t \leq 12\pi$$

3

$$\begin{aligned}x &= 0.2t \cos t \\y &= 0.1t \sin t \\z &= 0.1t\end{aligned} \quad 0 \leq t \leq 12\pi$$

1

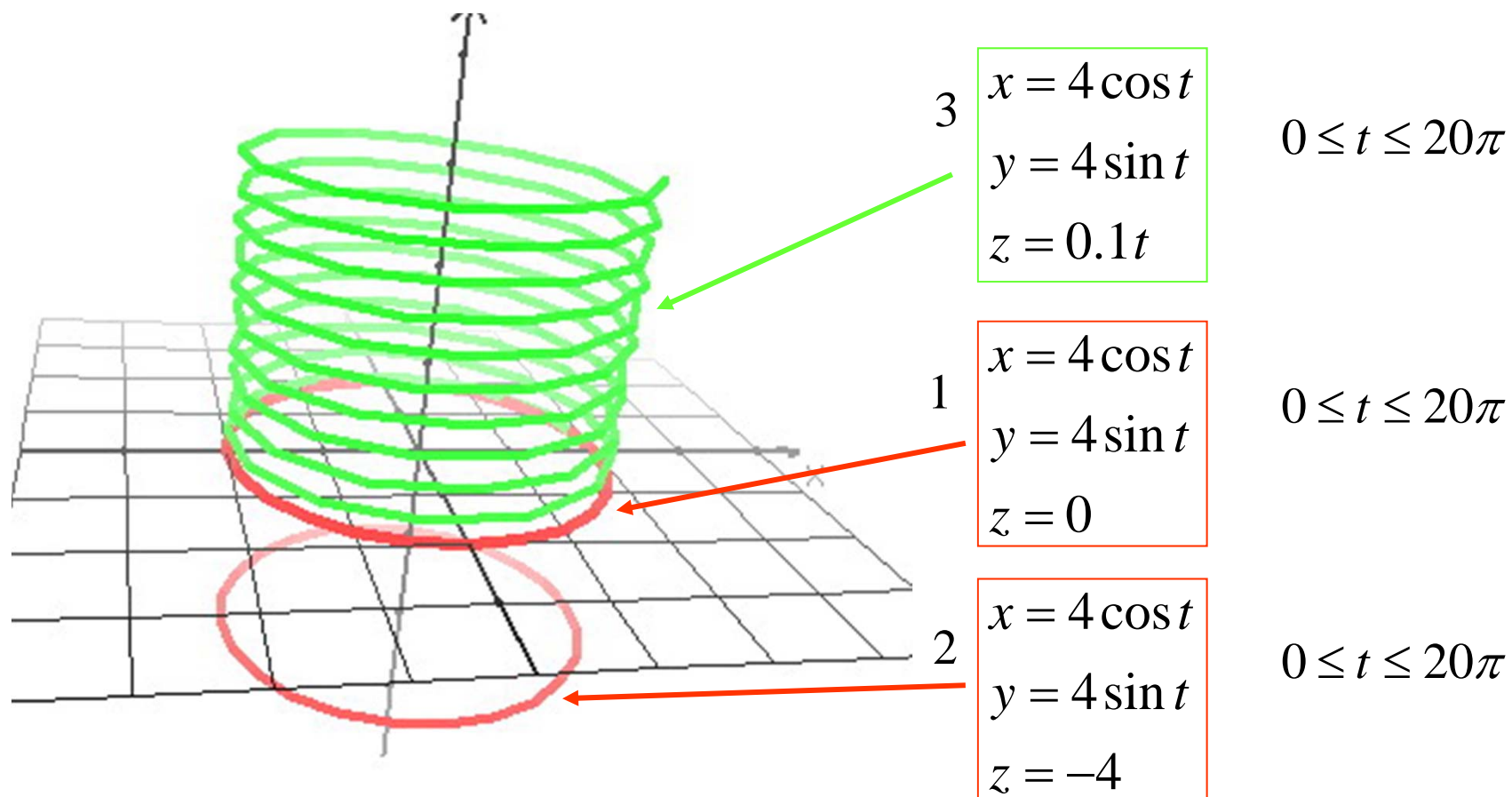
$$\begin{aligned}x &= 0.2t \cos t \\y &= 0.1t \sin t \\z &= 0\end{aligned} \quad 0 \leq t \leq 12\pi$$

2

$$\begin{aligned}x &= 0.2t \cos t \\y &= 0.1t \sin t \\z &= -5\end{aligned} \quad 0 \leq t \leq 12\pi$$

LINIJE (KRIVE) U PROSTORU

Primeri



LINIJE (KRIVE) U PROSTORU

$$X(t) = \alpha_1(t)$$

$$Y(t) = \alpha_2(t)$$

$$Z(t) = \alpha_3(t)$$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I \quad \text{— Kriva u prostoru}$$

$$\alpha_{xy}(t) = (\alpha_1(t), \alpha_2(t), 0), \quad t \in I \quad \text{— Projekcija krive na Oxy -ravan}$$

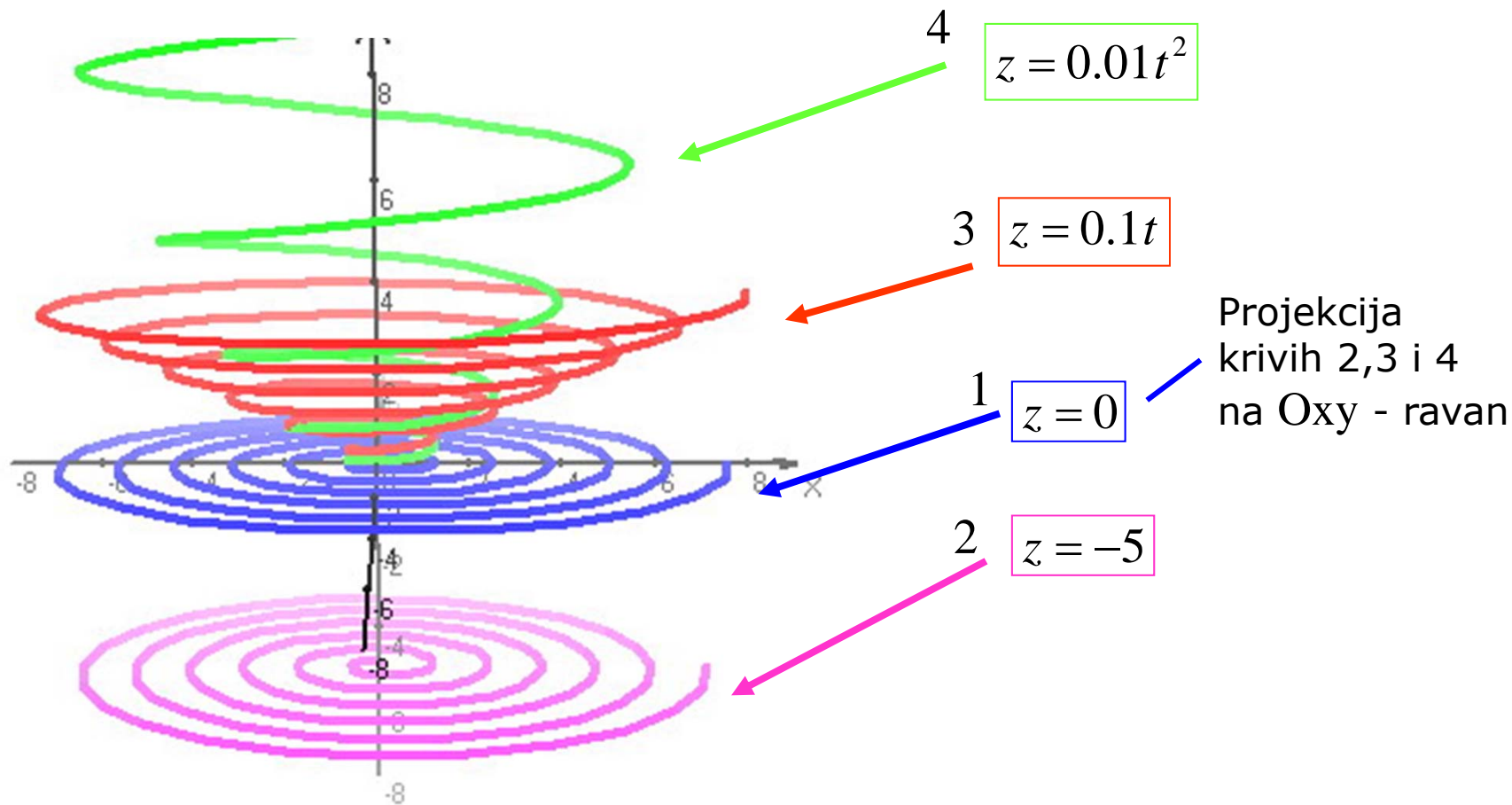
$$\alpha_{xz}(t) = (\alpha_1(t), 0, \alpha_3(t)), \quad t \in I \quad \text{— Projekcija krive na Oxz -ravan}$$

$$\alpha_{yz}(t) = (0, \alpha_2(t), \alpha_3(t)), \quad t \in I \quad \text{— Projekcija krive na Oyz -ravan}$$

LINIJE (KRIVE) U PROSTORU

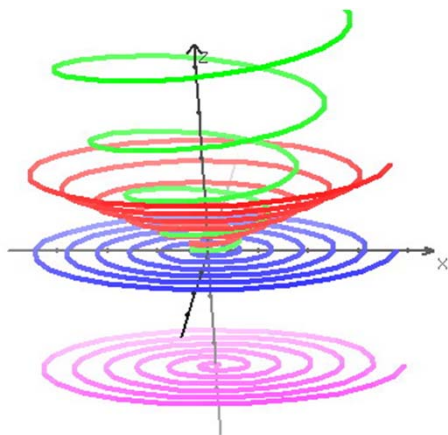
Primeri

$$\begin{aligned}x &= 0.2t \cos t \\y &= 0.1t \sin t\end{aligned} \quad 0 \leq t \leq 12\pi$$



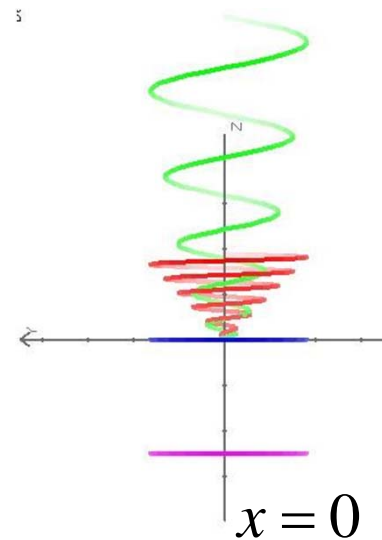
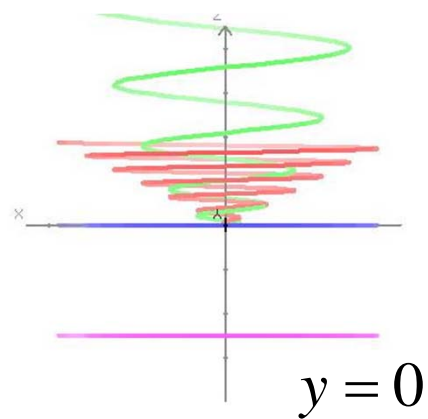
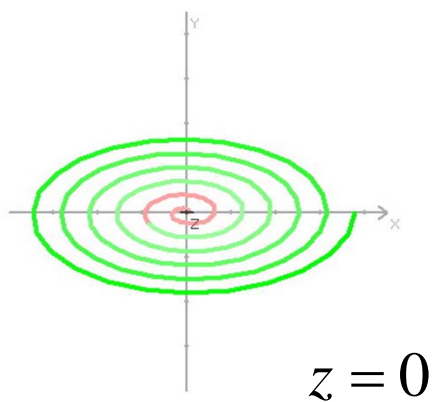
LINIJE (KRIVE) U PROSTORU

Primeri



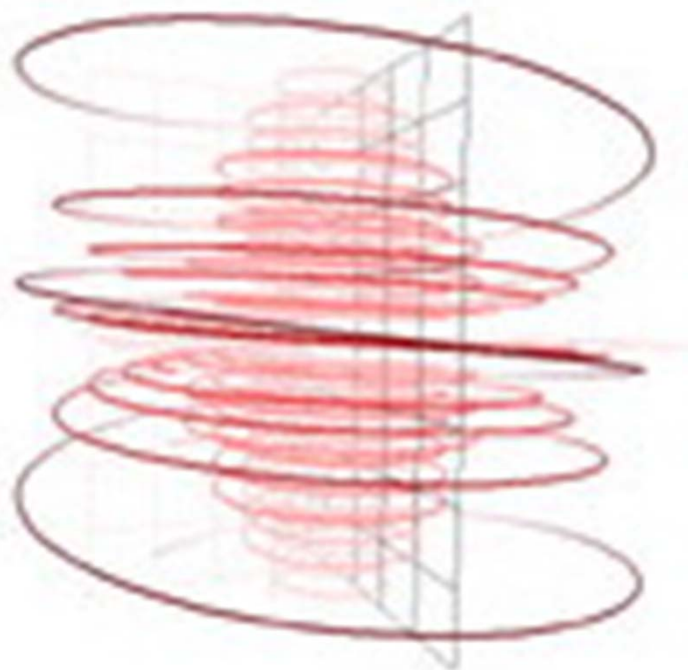
Projekcije na koordinatne ravni

Pogledi: odozgo, spreda, sa bočne strane



LINIJE (KRIVE) U PROSTORU

Primeri



Darko Kadvanj

$$x = a \sin t$$

$$y = a \cos t$$

$$z = t / (10 - a)$$

$$a = 1, 2, \dots, 10$$

$$-10 \leq t \leq 10$$

$$x = (1 - 10) \sin t$$

$$y = (1 - 10) \cos t$$

$$z = t / (10 - 1)$$

$$-10 \leq t \leq 10 \quad \text{Step } 0.1$$

Primer krivih u prostoru

Projekcija na Oxy - ravan su koncentrični krugovi:

$$x = (1 - 10) \sin t$$

$$y = (1 - 10) \cos t$$

$$z = 0$$

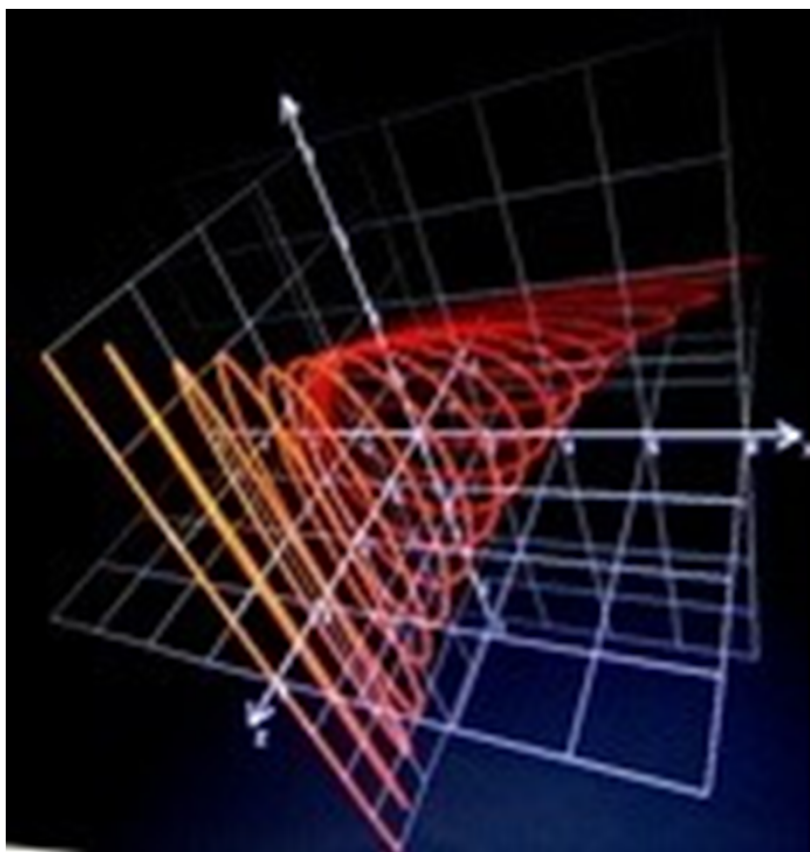
LINIJE (KRIVE) U PROSTORU

Primeri

$$x = a \cos t$$

$$y = b \sin t$$

$$z = c$$



Ivana Kula

$$x = (8 - 0) \cos t$$

$$y = (0 - 8) \sin t$$

$$z = (-8, \dots, 8)$$

$$0 \leq t \leq 2\pi \quad \text{Step } 0.3$$

17 elipsi u ravnima paralelnim sa Oxy ravni

Projekcija na Oxy - ravan su elipse:

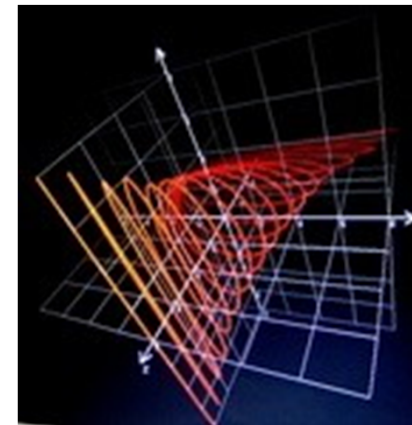
$$x = (8 - 0) \cos t$$

$$y = (0 - 8) \sin t$$

$$z = 0$$

LINIJE (KRIVE) U PROSTORU

Primeri



$$x = (8 - 0) \cos t$$

$$y = (0 - 8) \sin t$$

$$z = (-8, \dots, 8)$$

$$0 \leq t \leq 2\pi \quad \text{Step 0.3}$$

17 elipsi u ravnima
paralelnim sa Oxy ravni

$$x = a \cos t \quad a = 8, 7, \dots, -1, 0, 1, \dots, 7, 8$$

$$y = b \sin t \quad b = 0, 1, \dots, 7, 8, 7, \dots, 1, 0$$

$$z = c \quad c = -8, -7, \dots, 0, 1, \dots, 7, 8$$

$$x = a \cos t$$

$$y = (8 - a) \sin t \quad a = 8, 7, \dots, -1, 0, 1, \dots, 7, 8$$

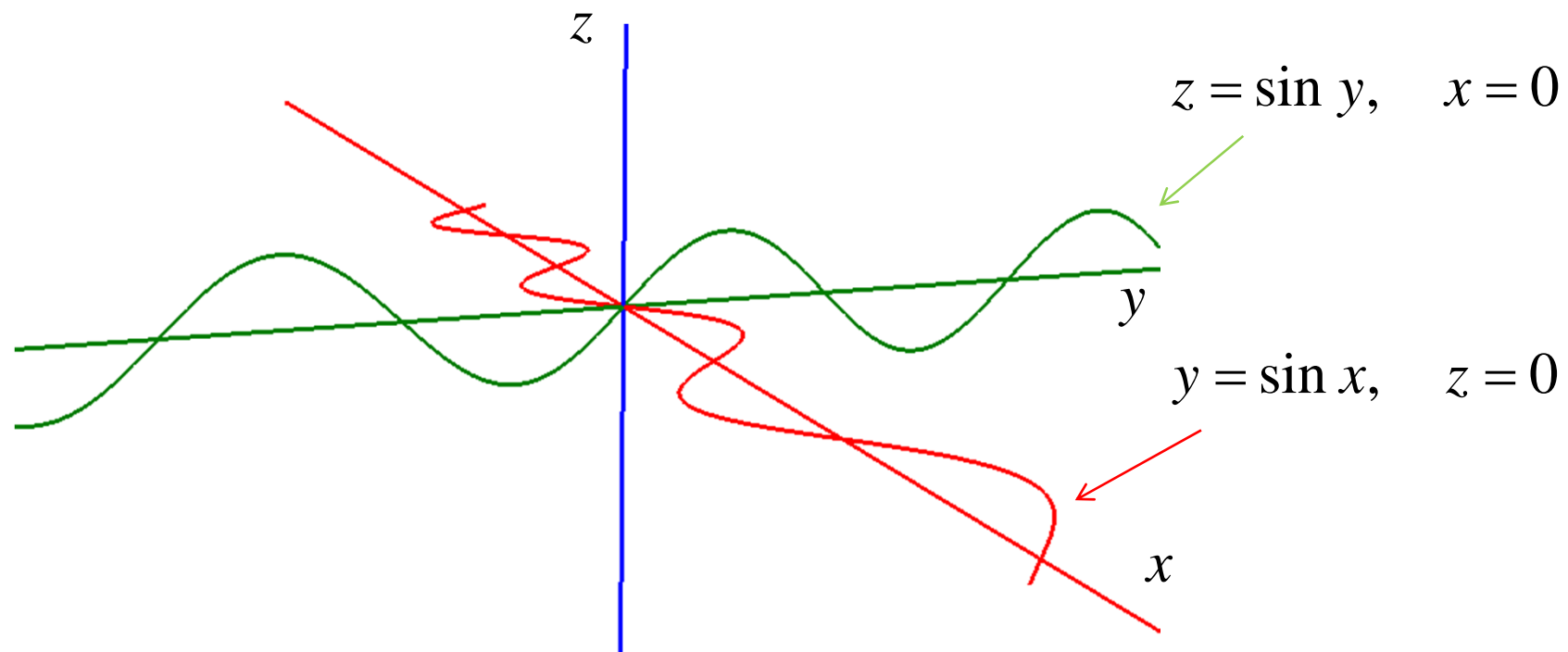
$$z = c \quad c = -8, -7, \dots, 0, 1, \dots, 7, 8$$

Ivana Kula

LINIJE (KRIVE) U PROSTORU

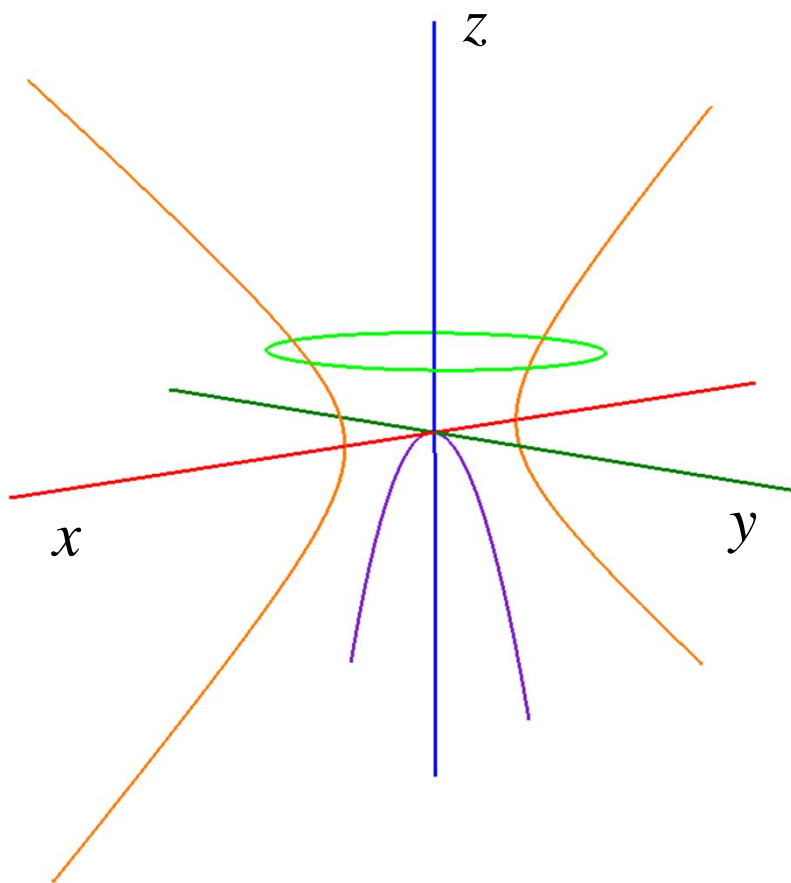
Ravne krive

Kriva u prostoru koja leži u nekoj ravni naziva se ravnom krivom.



LINIJE (KRIVE) U PROSTORU

Ravne krive



elipsa

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = c = \text{const}$$

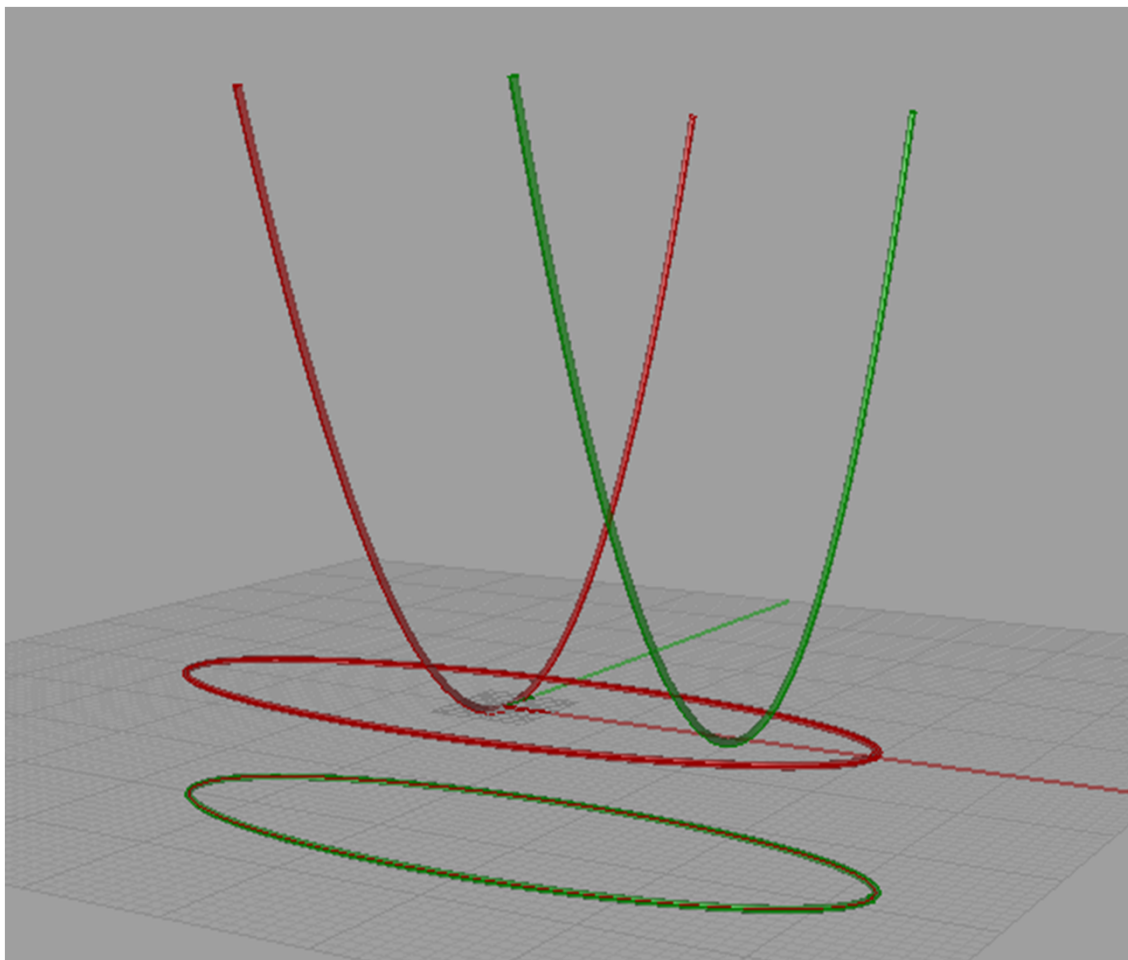
parabola

$$y^2 = 2pz, \quad x = 0$$

hiperbola

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1, \quad y = 0$$

LINIJE (KRIVE) U PROSTORU



parabola

$$y^2 = 2pz, \quad x = 0$$

parabola

$$y^2 = 2pz, \quad x = c \neq 0$$

elipsa

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = c = 0$$

elipsa

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = c \neq 0$$

LINIJE (KRIVE) U PROSTORU

Tangenta, normala, binormala

$$I = [a, b] \subset \mathbb{R}$$

Kriva u prostoru $\alpha : I \rightarrow \mathbb{R}^3$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I$$

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)), \quad t \in I$$

$$\alpha''(t) = (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t)), \quad t \in I$$

$$\alpha'''(t) = (\alpha_1'''(t), \alpha_2'''(t), \alpha_3'''(t)), \quad t \in I$$

LINIJE (KRIVE) U PROSTORU

Tangenta, normala, binormala

$$I = [a, b] \subset \mathbb{R} \quad \alpha : I \rightarrow \mathbb{R}^3$$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \quad |\alpha(t)| = \sqrt{\alpha_1^2(t) + \alpha_2^2(t) + \alpha_3^2(t)}$$

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)) \quad |\alpha'(t)| = \sqrt{\alpha_1'^2(t) + \alpha_2'^2(t) + \alpha_3'^2(t)}$$

$$\alpha''(t) = (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t)) \quad |\alpha''(t)| = \sqrt{\alpha_1''^2(t) + \alpha_2''^2(t) + \alpha_3''^2(t)}$$

$$\alpha'''(t) = (\alpha_1'''(t), \alpha_2'''(t), \alpha_3'''(t)) \quad |\alpha'''(t)| = \sqrt{\alpha_1'''^2(t) + \alpha_2'''^2(t) + \alpha_3'''^2(t)}$$

LINIJE (KRIVE) U PROSTORU

Tangenta, normala, binormala

Tangenta

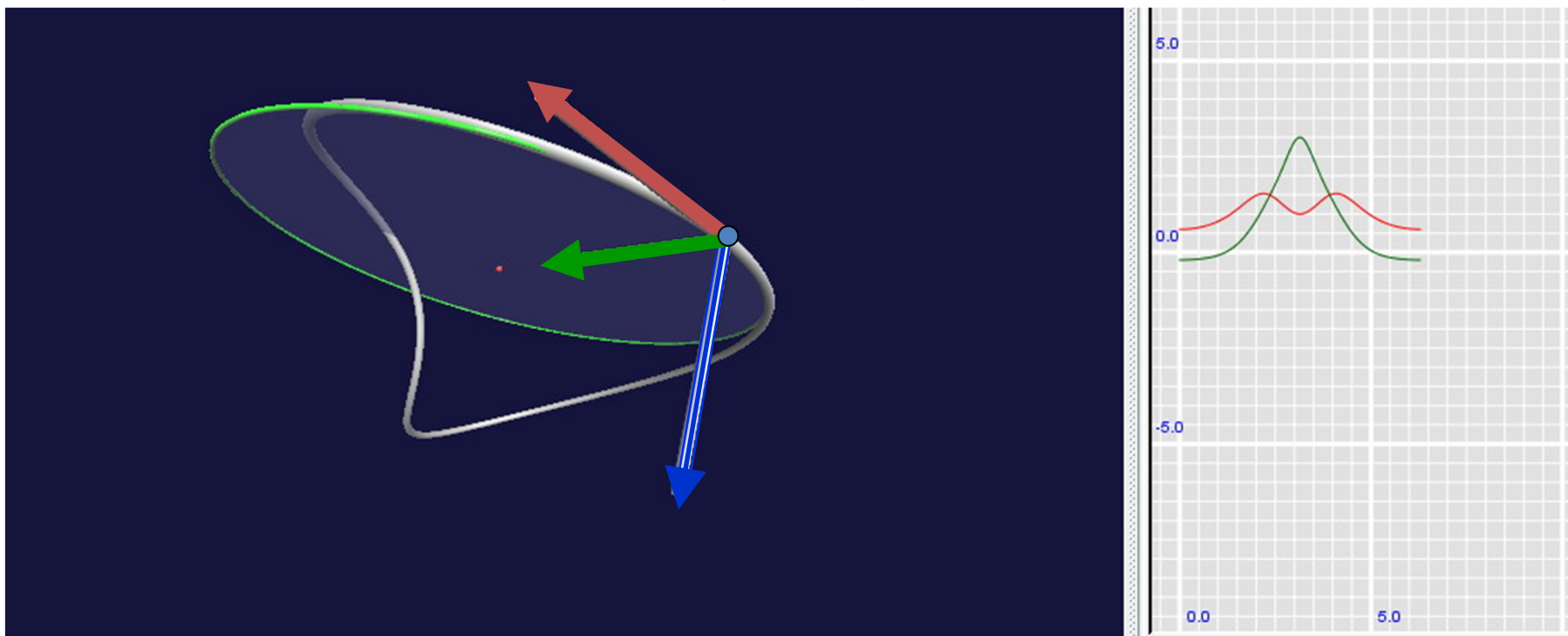
Normala

Binormala

$$\hat{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$$

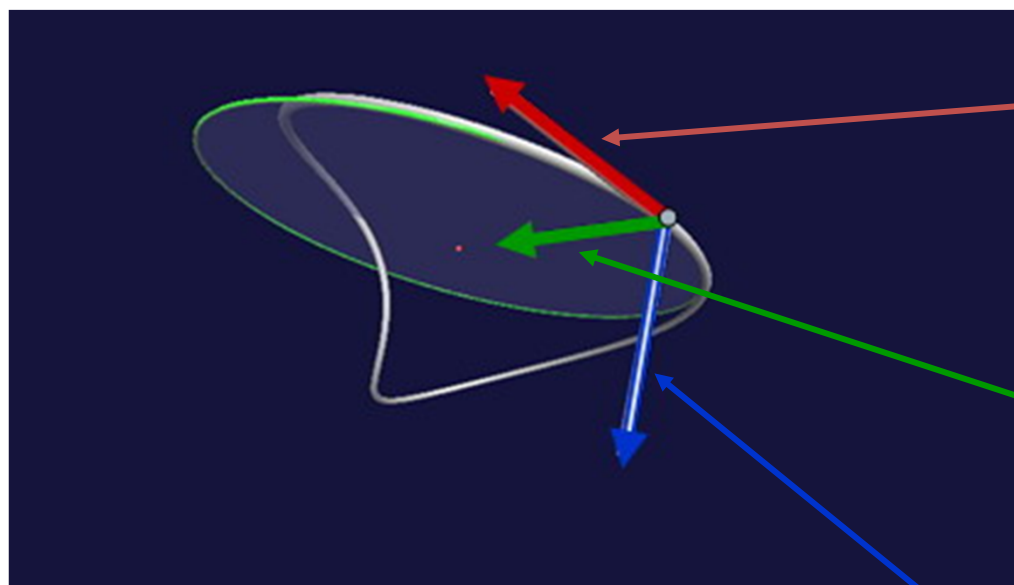
$$\hat{N}(t) = \frac{T'(t)}{|T'(t)|}$$

$$\hat{B}(t) = T(t) \times N(t)$$



LINIJE (KRIVE) U PROSTORU

Tangenta, normala, binormala



Jedinični vektor tangente:

$$\hat{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$$

Jedinični vektor normale:

$$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|}$$

Jedinični vektor binormale:

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$$

$\hat{T}(t), \hat{N}(t), \hat{B}(t)$ — Frene-ov trijedar

Pokretni prirodni trijedar

LINIJE (KRIVE) U PROSTORU

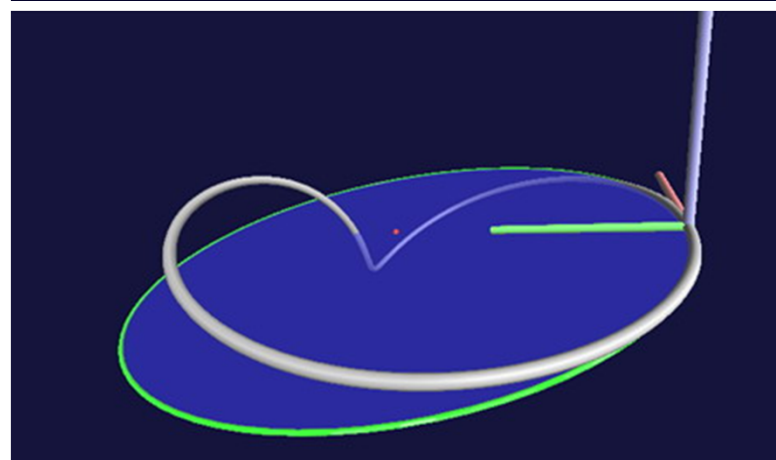
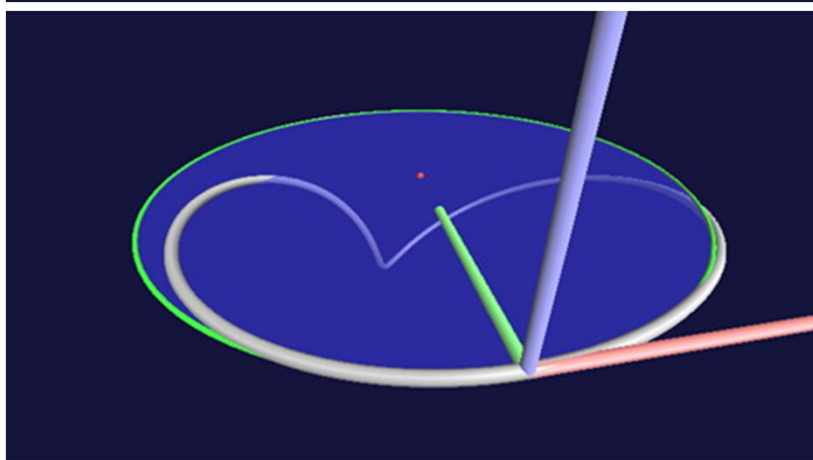
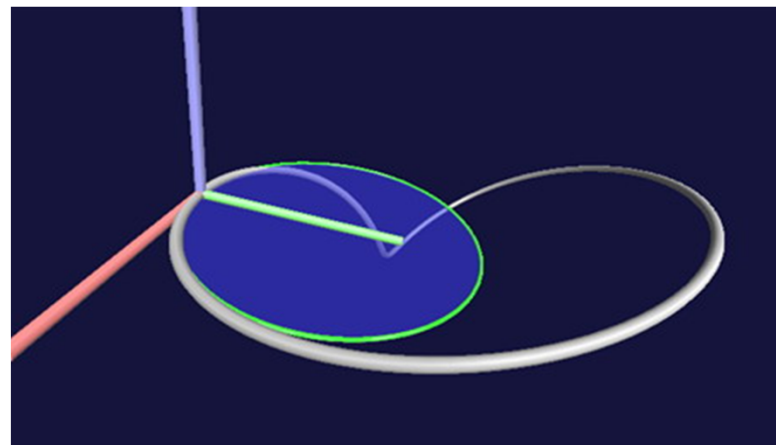
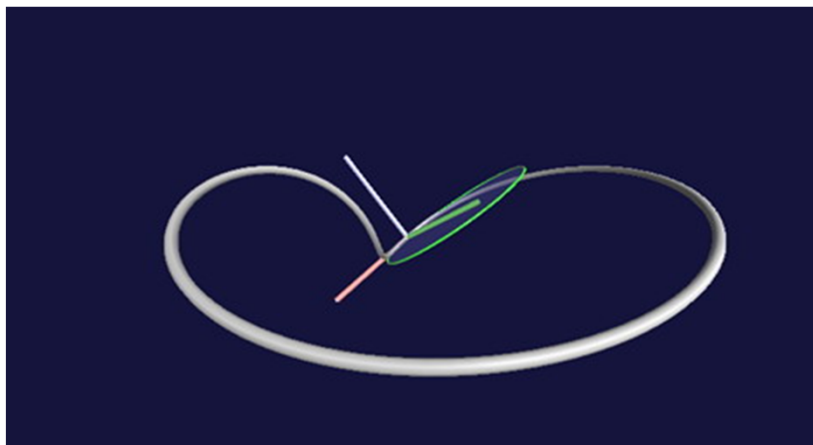
Primer

$$x = \cos t \cdot (1 + \cos t)$$

$$y = \sin t$$

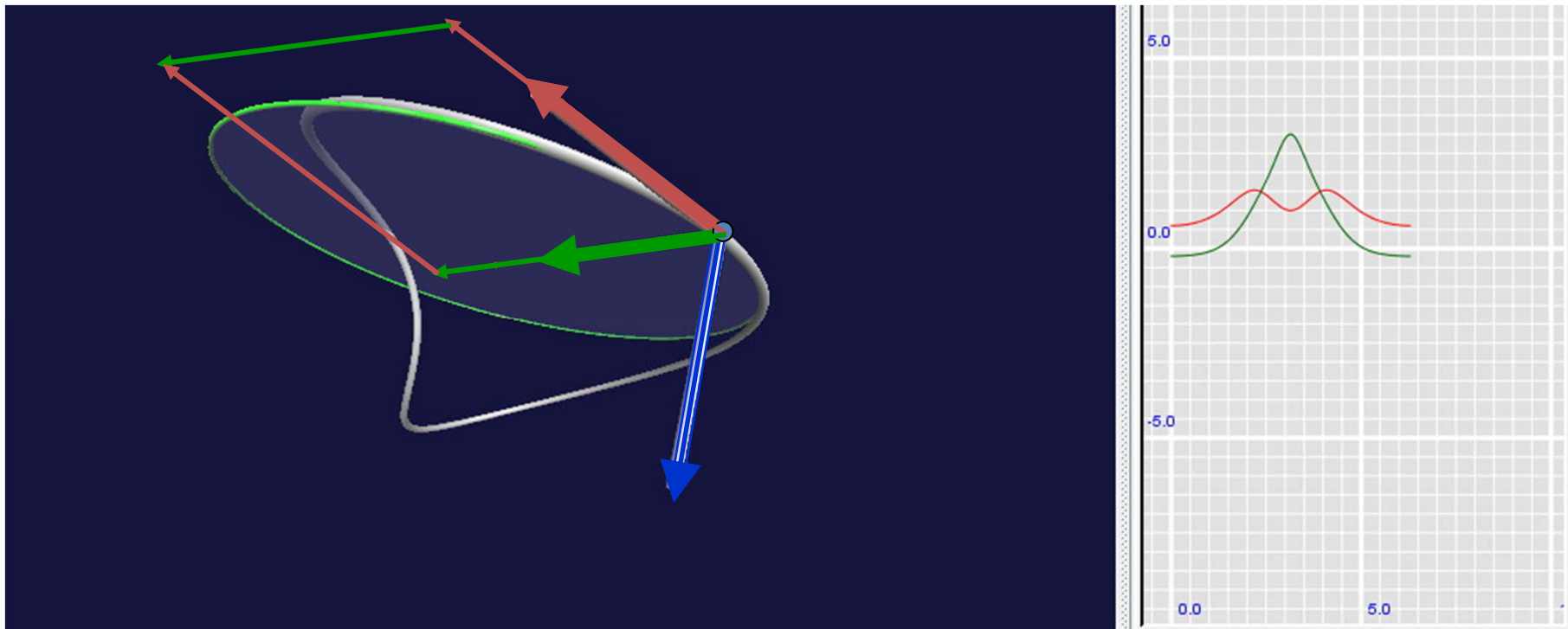
$$z = 1 + \cos t$$

$$0 \leq t \leq 2\pi$$



LINIJE (KRIVE) U PROSTORU

$T(t)$, $N(t)$ - oskulatorna ravan

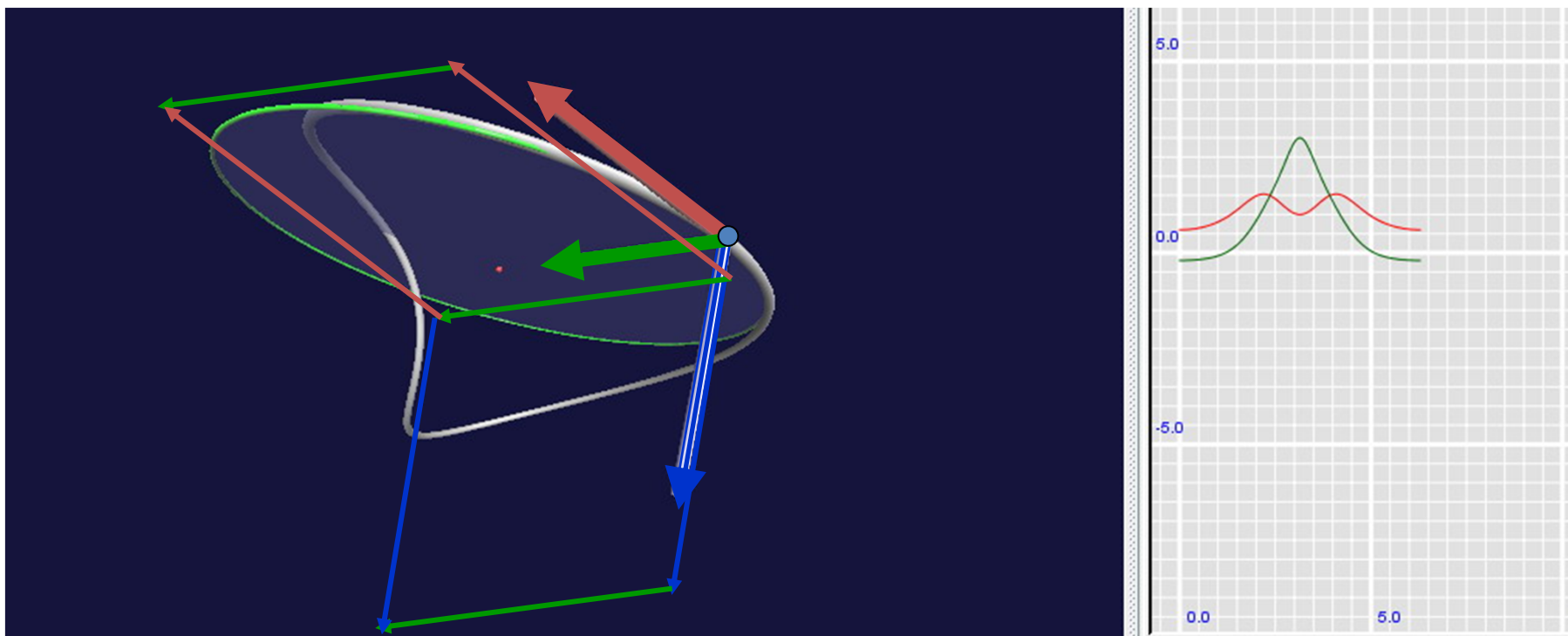


LINIJE (KRIVE) U PROSTORU

$T(t), N(t)$ - oskulatorna ravan

$N(t), B(t)$ - normalna ravan

$N(t), B(t)$ - normalna ravan



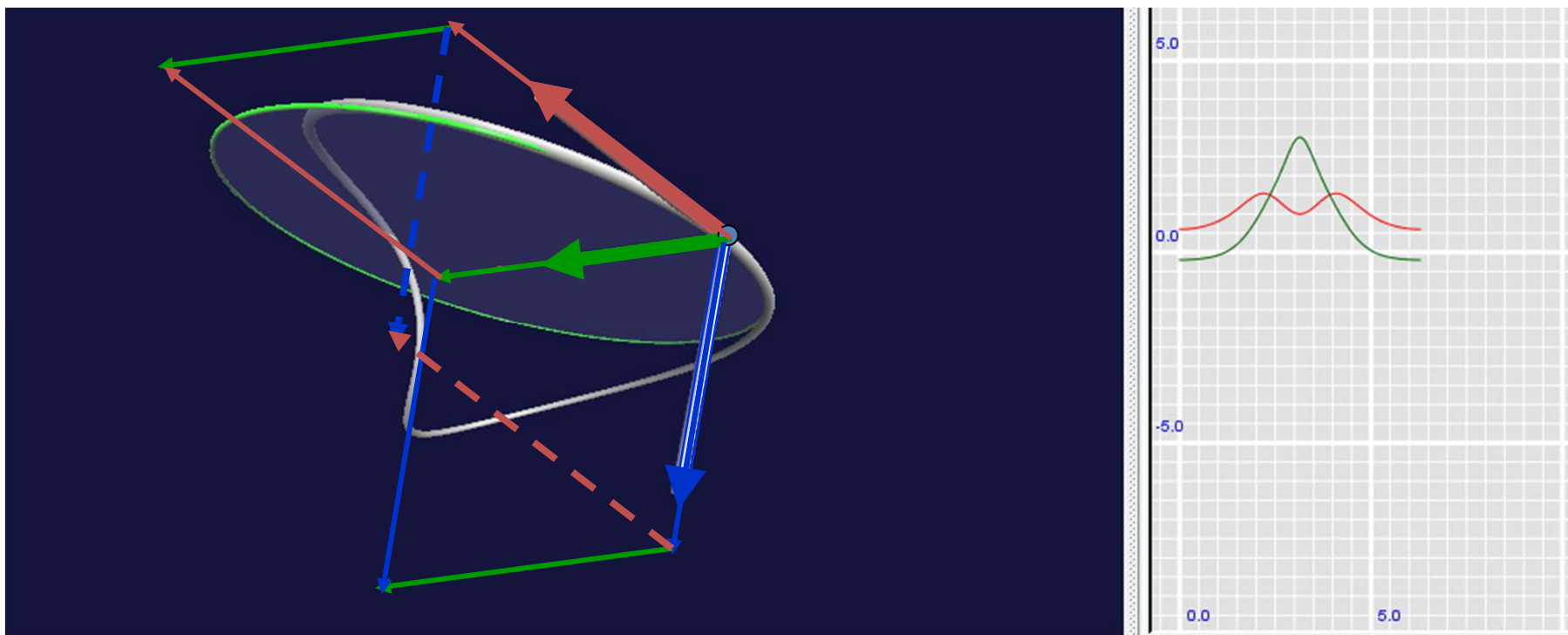
LINIJE (KRIVE) U PROSTORU

$T(t), N(t)$ - oskulatorna ravan

$N(t), B(t)$ - normalna ravan

$T(t), B(t)$ - rektifikaciona ravan

$T(t), B(t)$ - rektifikaciona ravan



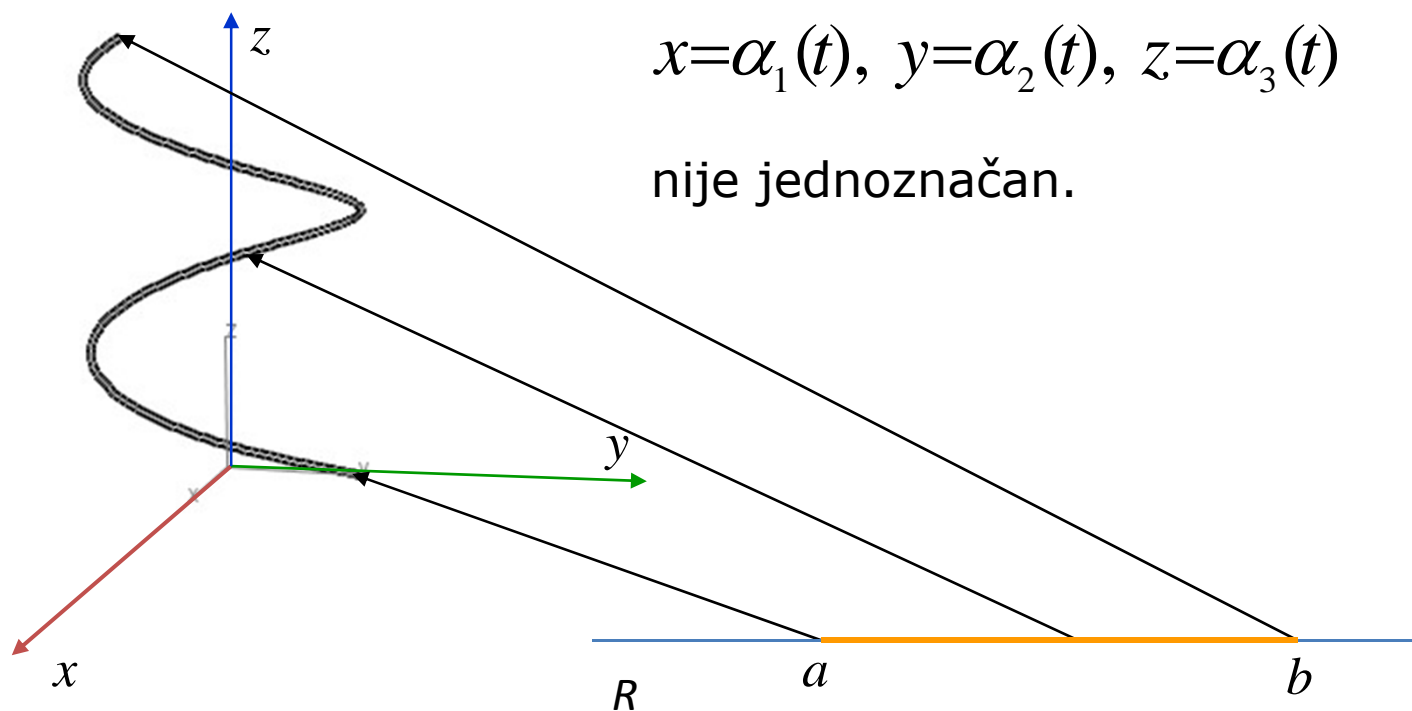
LINIJE (KRIVE) U PROSTORU

Reparametrizacija

Matematički prikaz krive linije C
u prostoru u parametarskom
obliku

$$x=\alpha_1(t), y=\alpha_2(t), z=\alpha_3(t) \quad a < t < b$$

nije jednoznačan.



LINIJE (KRIVE) U PROSTORU

Reparametrizacija

Ukoliko je

$$\varphi : J \rightarrow I \quad \varphi'(\tau) > 0$$

Diferencijabilna rastuća funkcija koja vrši preslikavanje nekog intervala $J = [c, d]$ neke parametarske prave τ na interval $I = [a, b]$ tada su

$$x = \alpha_1(\varphi(\tau)), y = \alpha_2(\varphi(\tau)), z = \alpha_3(\varphi(\tau)) \quad c < \tau < d$$

takodje parametarske jednačine iste krive C .

Kaže se da je izvršena reparametrizacija i nove jednačine su:

$$x = \beta_1(\tau), y = \beta_2(\tau), z = \beta_3(\tau) \quad c < \tau < d$$

LINIJE (KRIVE) U PROSTORU

Reparametrizacija

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I$$

$$I = [a, b] \quad J = [c, d]$$

$$\varphi: J \rightarrow I \quad t = \varphi(\tau), \quad \tau \in J$$

$$\alpha(t) = \alpha(\varphi(\tau)) = \beta(\tau), \quad \tau \in J$$

$$\beta'(\tau) = \alpha'(\varphi(\tau)) \cdot \varphi'(\tau), \quad \tau \in J$$

LINIJE (KRIVE) U PROSTORU

Reparametrizacija

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I \qquad t = \varphi(\tau), \quad \tau \in J$$

$$\beta(\tau) = (\beta_1(\tau), \beta_2(\tau), \beta_3(\tau)), \quad \tau \in J$$

$$X(t) = \alpha_1(t)$$

$$Y(t) = \alpha_2(t) \quad a \leq t \leq b$$

$$Z(t) = \alpha_3(t)$$

$$X(t) = \beta_1(\tau)$$

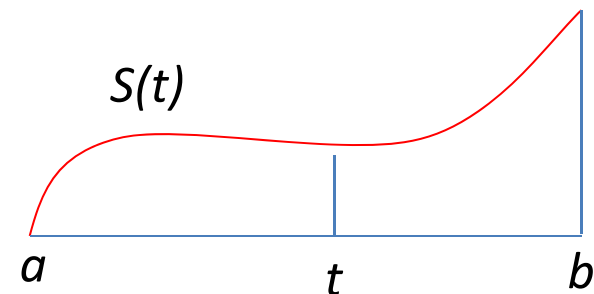
$$Y(t) = \beta_2(\tau) \quad c \leq \tau \leq d$$

$$Z(t) = \beta_3(\tau)$$

$$t = \varphi(\tau), \quad \tau \in J$$

LINIJE (KRIVE) U PROSTORU

Prirodna reparametrizacija



$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I \quad I = [a, b]$$

$$s'(t) = \sqrt{\alpha_1'^2(t) + \alpha_2'^2(t) + \alpha_3'^2(t)}$$

$$s(t) = \int_a^t s'(t) dt = \int_a^t \sqrt{\alpha_1'^2(t) + \alpha_2'^2(t) + \alpha_3'^2(t)} dt$$

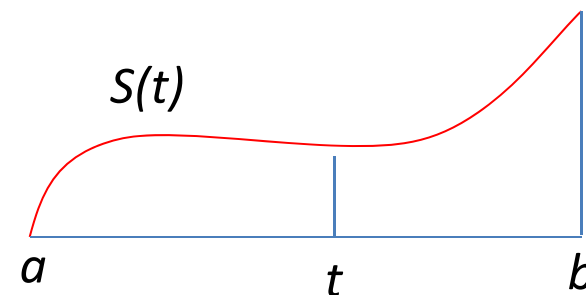
$$s(a) = 0 \quad I = [a, b] \quad J = [0, s(b)]$$

$$\varphi : I \rightarrow J \quad \varphi^{-1} : J \rightarrow I \quad s = \varphi(t) \quad t = \varphi^{-1}(s)$$

$$t \in I, \quad s \in J$$

LINIJE (KRIVE) U PROSTORU

Prirodna reparametrizacija



$$s(a) = 0 \quad I = [a, b] \quad J = [0, s(b)]$$

$$\varphi : I \rightarrow J \quad \varphi^{-1} : J \rightarrow I \quad s = \varphi(t) \quad t = \varphi^{-1}(s)$$
$$t \in I, \quad s \in J$$

$$\alpha(t) = \alpha(\varphi^{-1}(s)) = \gamma(s)$$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I$$

$$\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)), \quad s \in J$$

LINIJE (KRIVE) U PROSTORU

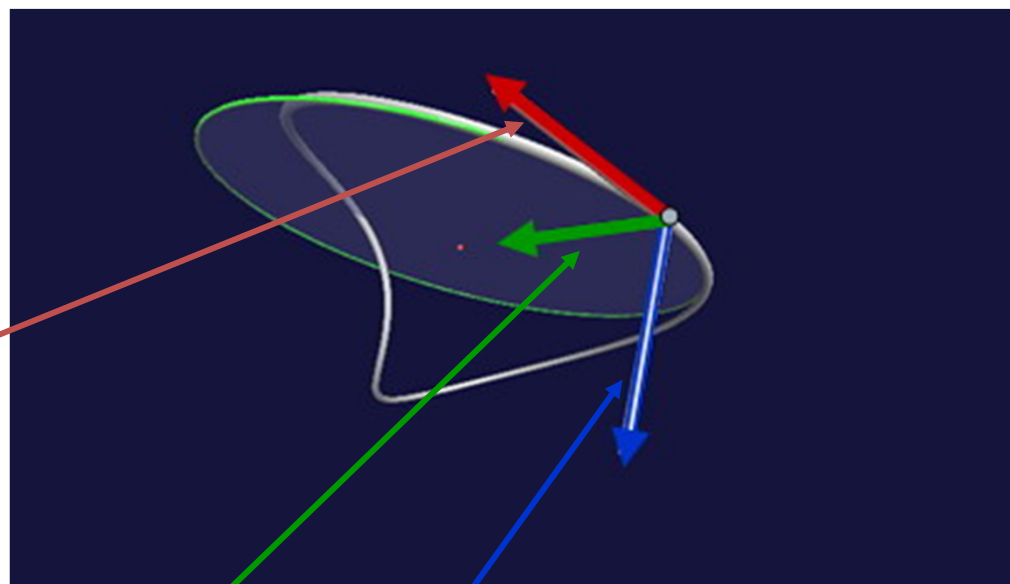
Prirodna reparametrizacija

$$|\gamma'(s)| = 1$$

$$\hat{T}(s) = \gamma'(s)$$

$$\hat{N}(s) = \frac{\hat{T}'(s)}{|\hat{T}'(s)|} = \frac{\gamma''(s)}{|\gamma''(s)|}$$

$$\hat{B}(s) = T(s) \times N(s)$$



LINIJE (KRIVE) U PROSTORU

Prirodna reparametrizacija

Tangenta

Normala

Binormala

Jedinični vektori tangente, normale i binormale su invarijantni u odnosu na parametrizaciju:

$$\hat{T}(t) (= \hat{T}(\tau) = \hat{T}(s)) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\beta'(\tau)}{|\beta'(\tau)|} = \gamma'(s)$$

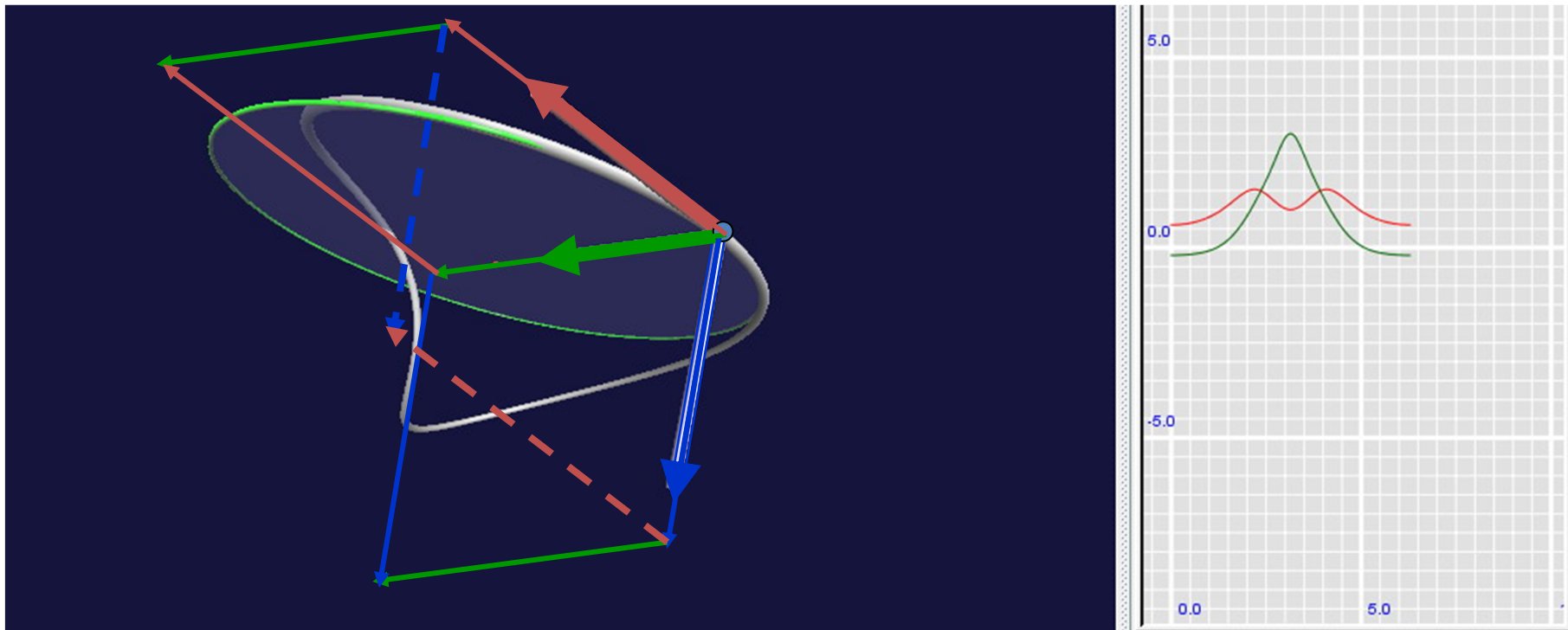
$$\hat{N}(t) (= \hat{N}(\tau) = \hat{N}(s)) = \frac{T'(t)}{|T'(t)|} = \frac{T'(\tau)}{|T'(\tau)|} = \frac{\gamma''(s)}{|\gamma''(s)|}$$

$$\hat{B}(t) (= \hat{B}(\tau) = \hat{B}(s)) = \hat{T}(t) \times \hat{N}(t) = \frac{\gamma'(s) \times \gamma''(s)}{|\gamma''(s)|}$$

LINIJE (KRIVE) U PROSTORU

Krivina

$$K(s) = \left| \frac{d\hat{T}(s)}{ds} \right| = |\hat{T}'(s)| = |\gamma''(s)|$$



LINIJE (KRIVE) U PROSTORU

Krivina

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in I$$

$$X(t) = \alpha_1(t)$$

$$Y(t) = \alpha_2(t) \quad a \leq t \leq b$$

$$Z(t) = \alpha_3(t)$$

$$K(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}$$

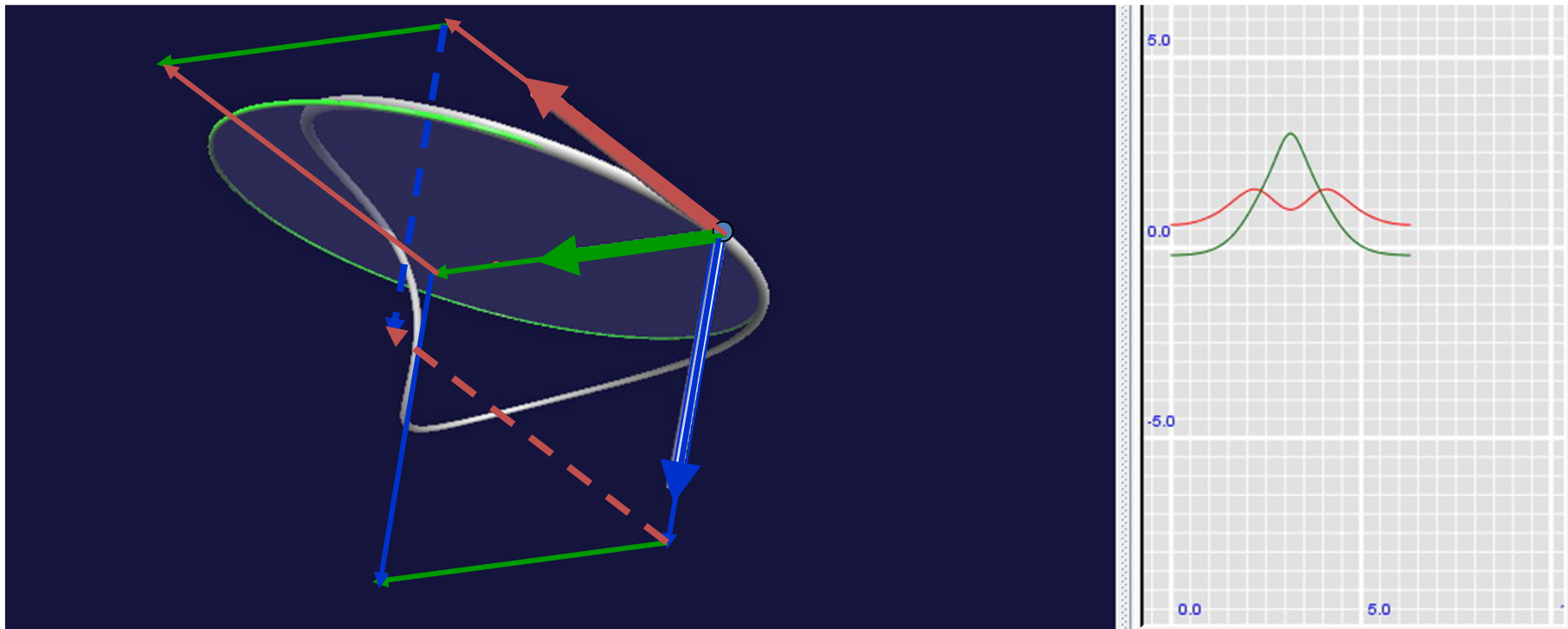
$$K(t) = \frac{\sqrt{(\alpha_3''\alpha_2' - \alpha_2''\alpha_3')^2 + (\alpha_2''\alpha_1' - \alpha_1''\alpha_2')^2 + (\alpha_1''\alpha_3' - \alpha_3''\alpha_1')^2}}{(\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2)^{\frac{3}{2}}}$$

LINIJE (KRIVE) U PROSTORU

Krivina

Poluprečnik krivine: $R = \frac{1}{K}$

Krug krivine



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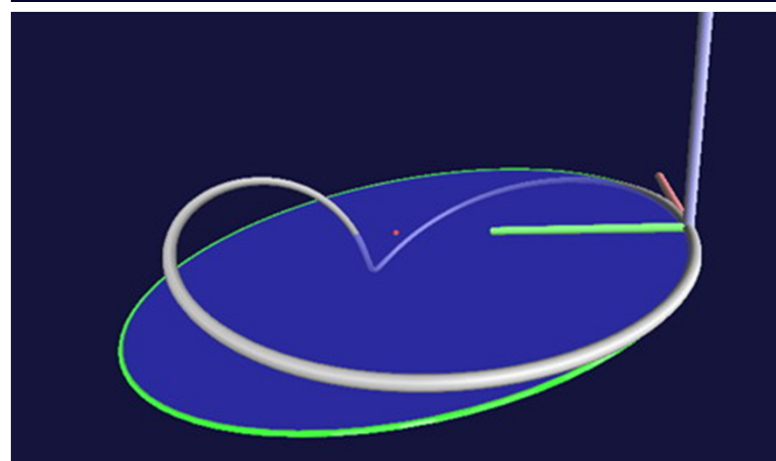
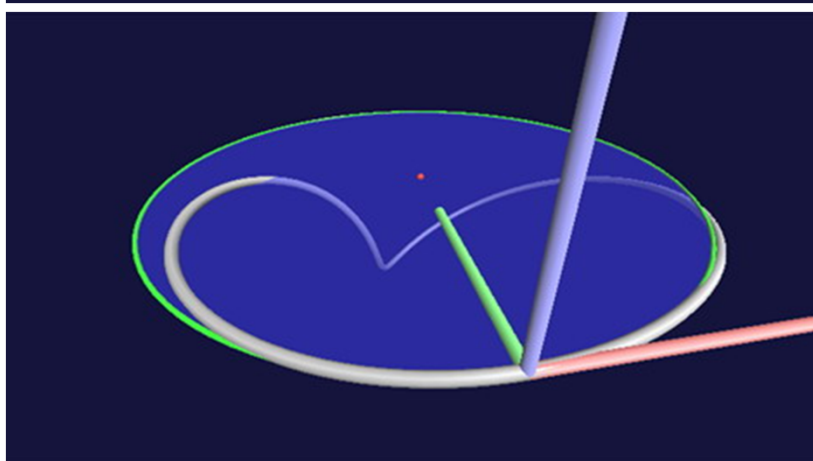
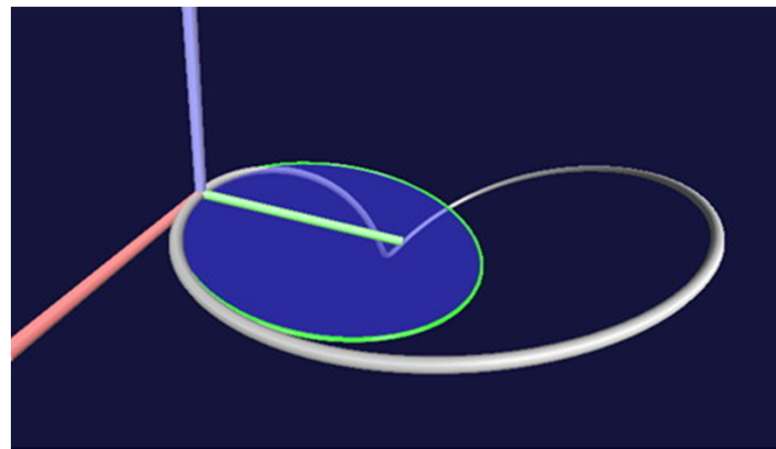
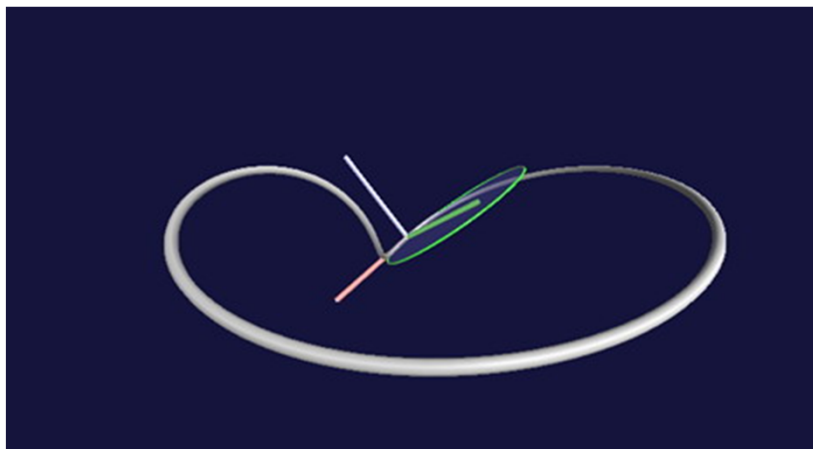
Krivina - primer

$$x = \cos t \cdot (1 + \cos t)$$

$$y = \sin t$$

$$z = 1 + \cos t$$

$$0 \leq t \leq 2\pi$$



LINIJE (KRIVE) U PROSTORU

Geometrijsko mesto centara krugova krivina neke krive je evoluta te krive. Sama ta kriva, za svoju evolutu je evolventa.

