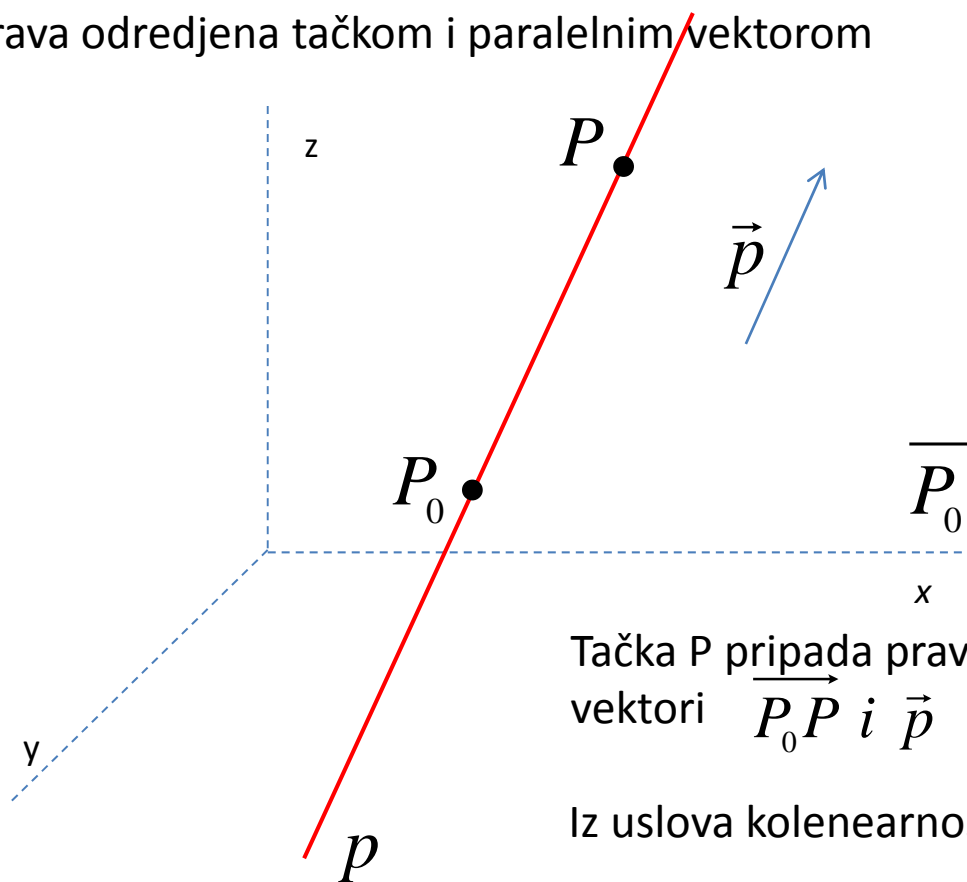


JEDNAČINE PRAVE - KANONSKI OBLIK JEDNAČINA PRAVE  
Prava određena tačkom i paralelnim vektorom



$p$  :

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

Tačka  $P$  pripada pravoj  $p$  ako i samo ako su vektori  $\overrightarrow{P_0P}$  i  $\vec{p}$  kolimearni.

Iz uslova kolimearnosti slede jednačine prave:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

PRIMER

Napisati jednačine prave koja sadrži tačku  $P_0(1, -2, 3)$

i paralelna je vektoru  $\vec{p} = (5, 2, -3)$ .

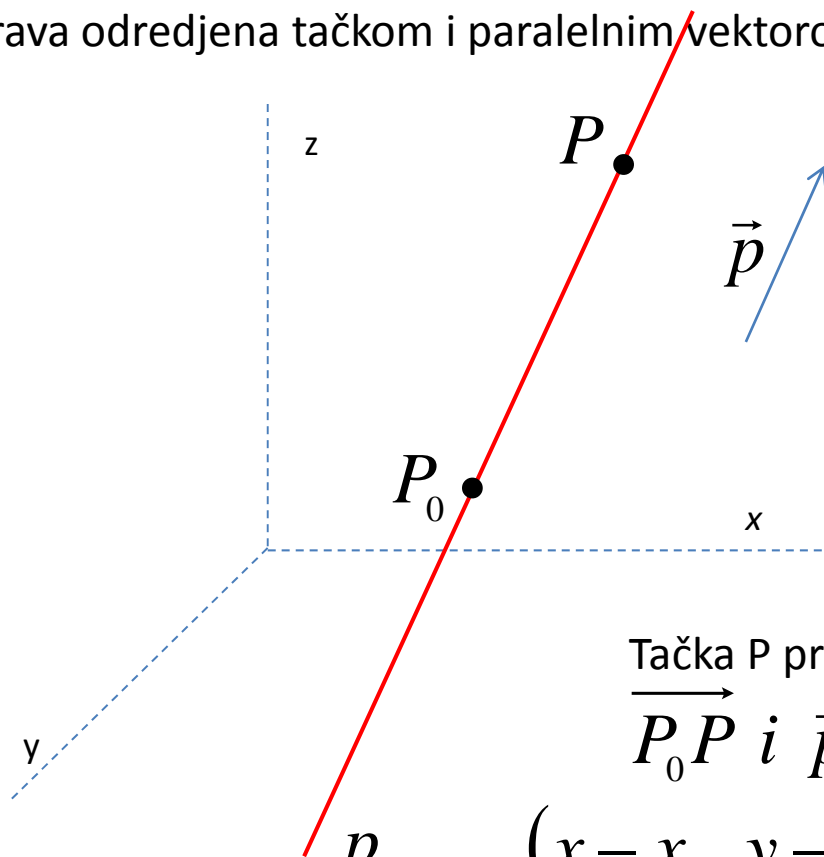
$$P: \quad P_0(1, -2, 3) \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
$$\vec{p} = (5, -3, 2)$$

---

$$\frac{x - 1}{5} = \frac{y + 2}{-3} = \frac{z - 3}{2}$$

## PARAMETARSKÉ JEDNAČINE PRAVE

Prava odredjena tačkou i paralelnim vektorom



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

Tačka  $P$  pripada pravoj ako i samo ako su vektori

$$\overrightarrow{P_0P} \text{ i } \vec{p} \text{ kolinearni: } \overrightarrow{P_0P} = t \cdot \vec{p}, \quad t \in \mathbb{R}$$

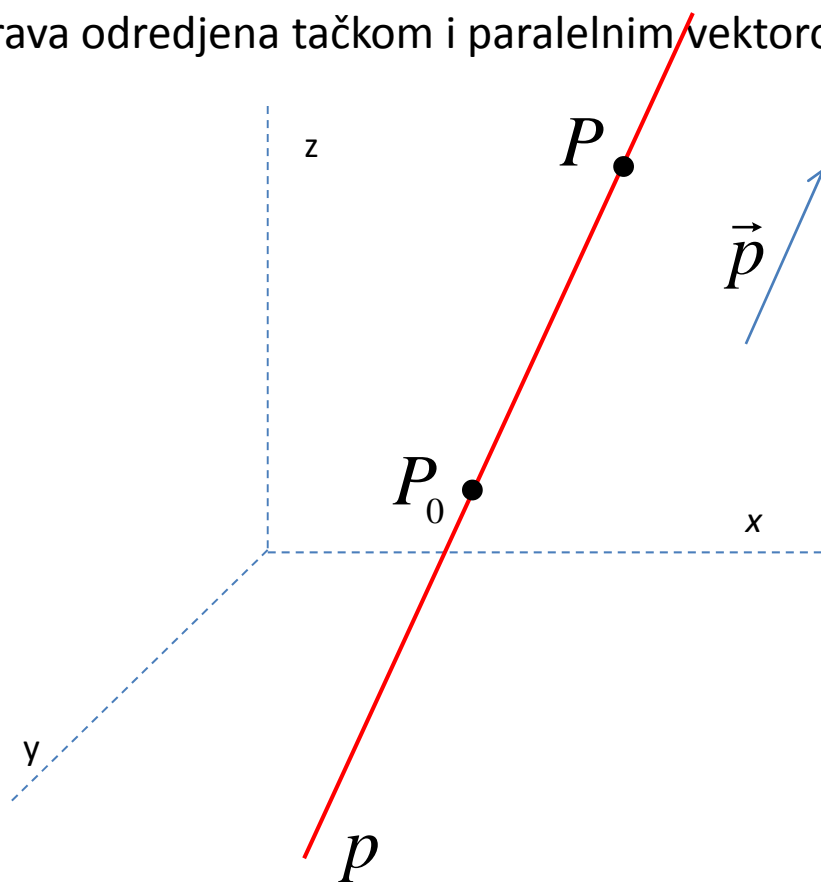
$$p \quad (x - x_0, y - y_0, z - z_0) = t(a, b, c), \quad t \in \mathbb{R}$$

$$x - x_0 = at, \quad y - y_0 = bt, \quad z - z_0 = ct, \quad t \in \mathbb{R}$$

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad t \in \mathbb{R}$$

## PARAMETARSKE JEDNAČINE PRAVE

Prava određena tačkom i paralelnim vektorom



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

---

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

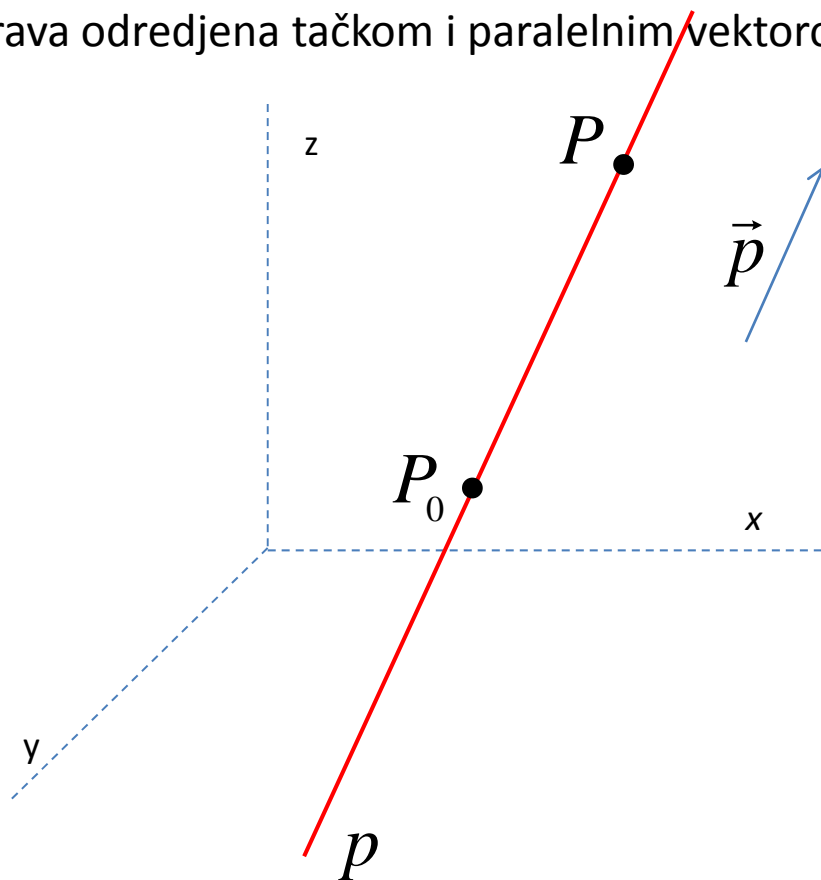
$$x = x_0 + at$$

$$y = y_0 + bt \quad t \in \mathbb{R}$$

$$z = z_0 + ct$$

## PARAMETARSKE JEDNAČINE PRAVE

Prava određena tačkom i paralelnim vektorom



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

$$x = x_0 + at$$

$$y = y_0 + bt \quad t \in \mathbb{R}$$

$$z = z_0 + ct$$

PRIMER

Napisati parametarske jednačine prave koja sadrži tačku  $P_0(1, -2, 3)$

i paralelna je vektoru  $\vec{p} = (5, -3, 2)$ .

$$p: \begin{aligned} &P_0(x_0, y_0, z_0) \\ &\vec{p} = (a, b, c) \end{aligned}$$

---

$$x = x_0 + at$$

$$y = y_0 + bt \quad t \in R$$

$$z = z_0 + ct$$

$$p: \begin{aligned} &P_0(1, -2, 3) \\ &\vec{p} = (5, -3, 2) \end{aligned}$$

---

$$x = 1 + 5t$$

$$y = -2 - 3t \quad t \in R$$

$$z = 3 + 2t$$

PRIMER\*

Napisati parametarske jednačine prave koja sadrži tačku  $P_0(1, -2, 3)$

i paralelna je vektoru  $\vec{p} = (5, -3, 2)$ .

$$p: P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

---


$$x = x_0 + at$$

$$y = y_0 + bt \quad t \in R$$

$$z = z_0 + ct$$

$$p: P_0(1, -2, 3)$$

$$\vec{p} = (5, -3, 2)$$

$$\frac{x - 1}{5} = \frac{y + 2}{-3} = \frac{z - 3}{2} = t$$

---

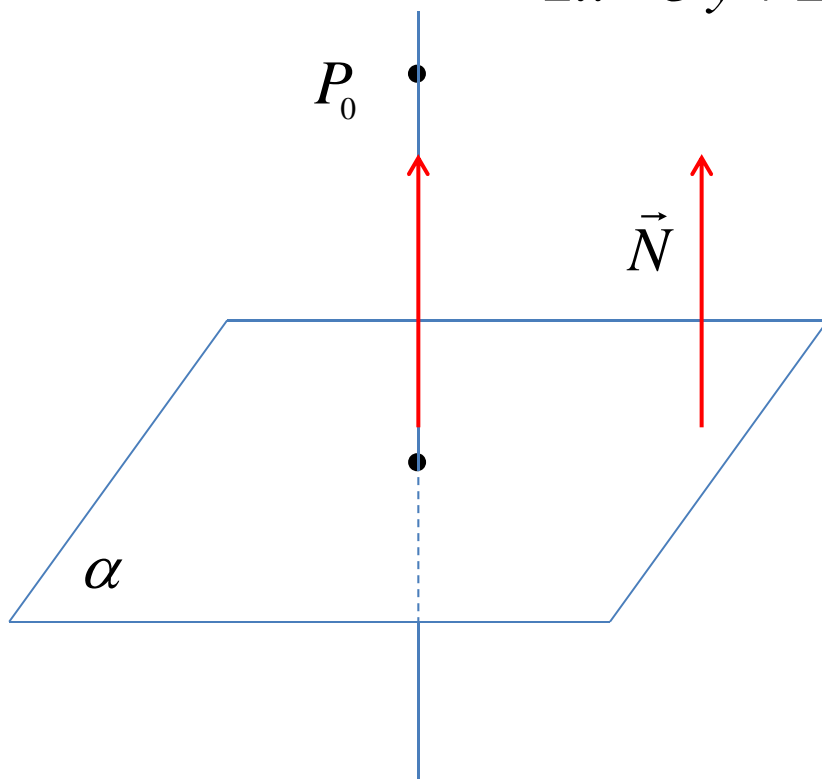

$$x = 1 + 5t$$

$$y = -2 - 3t \quad t \in R$$

$$z = 3 + 2t$$

PRIMER

Napisati jednačine prave koja sadrži tačku  $P_0(-1, 2, -3)$   
i normalna je na ravan  $-2x - 3y + 2z + 4 = 0$ .



$p:$

$$P_0(-1, 2, -3)$$

$$\vec{p} = \vec{N} = (-2, -3, 2)$$

Kanonske jednačine

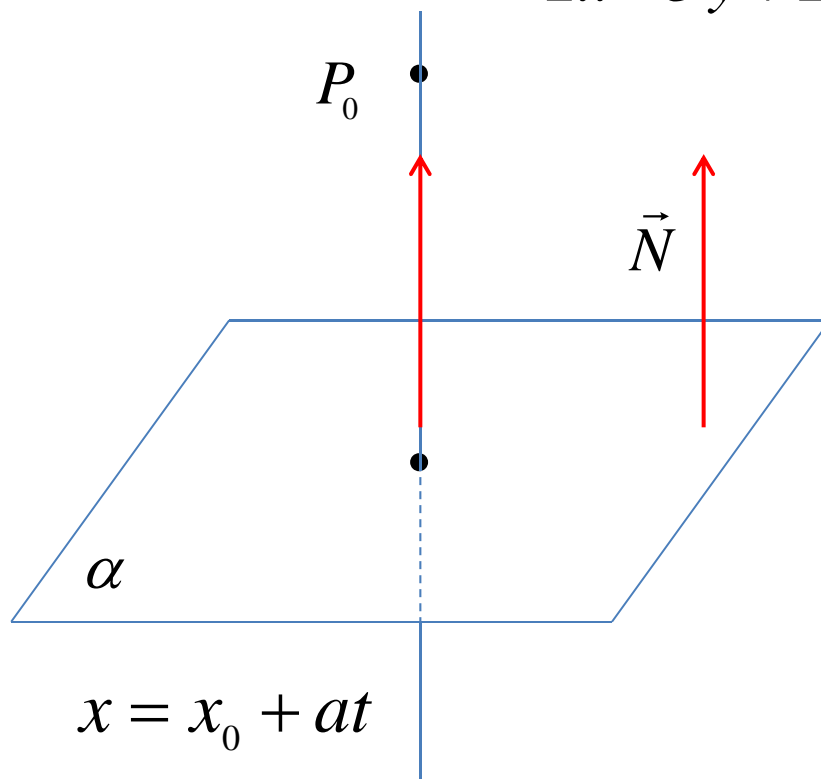
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\frac{x + 1}{-2} = \frac{y - 2}{-3} = \frac{z + 3}{2}$$



PRIMER\*

Napisati parametarske jednačine prave koja sadrži tačku  $P_0(-1, 2, -3)$  i normalna je na ravan  $-2x - 3y + 2z + 4 = 0$ .



$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \quad t \in \mathbb{R}$$

$p$ :

$$P_0(-1, 2, -3)$$

$$\vec{p} = \vec{N} = (-2, -3, 2)$$

---


$$\frac{x+1}{-2} = \frac{y-2}{-3} = \frac{z+3}{2} = t$$

Parametarske jednačine

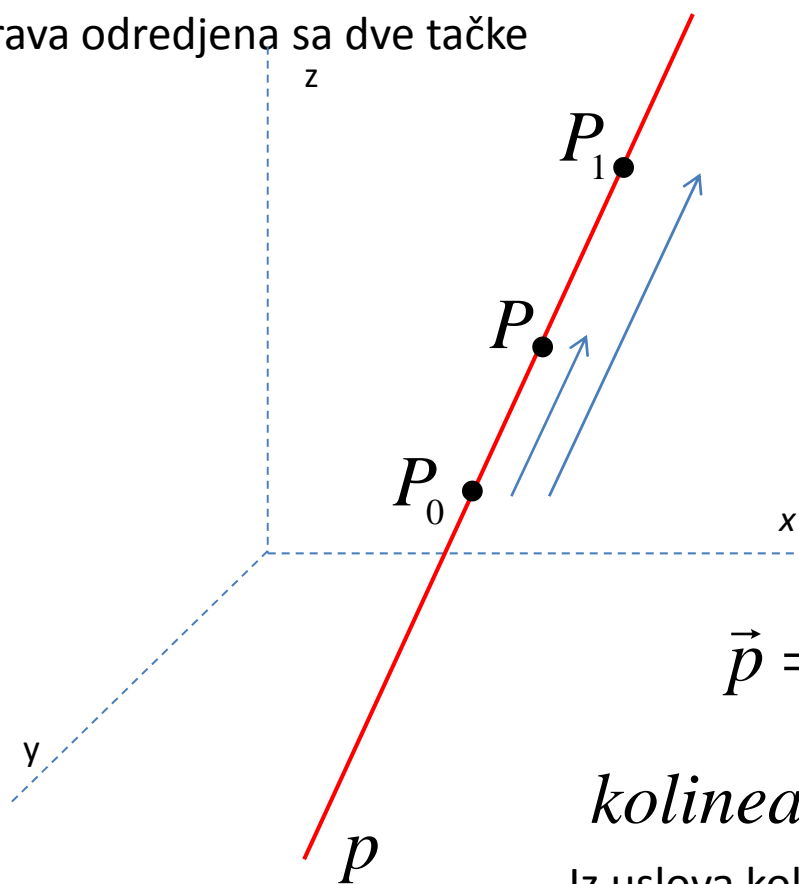
$$x = -1 - 2t$$

$$y = 2 - 3t \quad t \in \mathbb{R}$$

$$z = -3 + 2t$$

## JEDNAČINE PRAVE

Prava određena sa dve tačke



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

$$\vec{p} = \overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

*kolinearnost*  $\overrightarrow{P_0P}$  i  $\overrightarrow{P_0P_1}$

Iz uslova koleniarnosti slede jednačine prave:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

PRIMER

Napisati jednačine prave određene sa dve tacke  $P_0(1, -2, 3)$  i  $P_1(3, 4, 1)$ .

$$\vec{p} = \overrightarrow{P_0P_1} = (3 - 1, 4 + 2, 1 - 3) = (2, 6, -2)$$

$$\vec{p} = (2, 6, -2)$$

---

$$P: \quad P_0(1, -2, 3) \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
$$\vec{p} = (2, 6, -2)$$

---

$$\frac{x - 1}{2} = \frac{y + 2}{6} = \frac{z - 3}{-2} \quad | \cdot 2 \Rightarrow x - 1 = \frac{y + 2}{3} = -(z - 3)$$

PRIMER\*

Napisati jednačine prave određene sa dve tacke  $P_0(1, -2, 3)$  i  $P_1(3, 4, 1)$ .

$$\vec{p} = \overrightarrow{P_0P_1} = (3 - 1, 4 + 2, 1 - 3) = (2, 6, -2)$$

$$\vec{p} = (2, 6, -2) \quad \vec{p} = (1, 3, -1)$$

---

$$P: \quad P_0(1, -2, 3) \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
$$\vec{p} = (1, 3, -1)$$

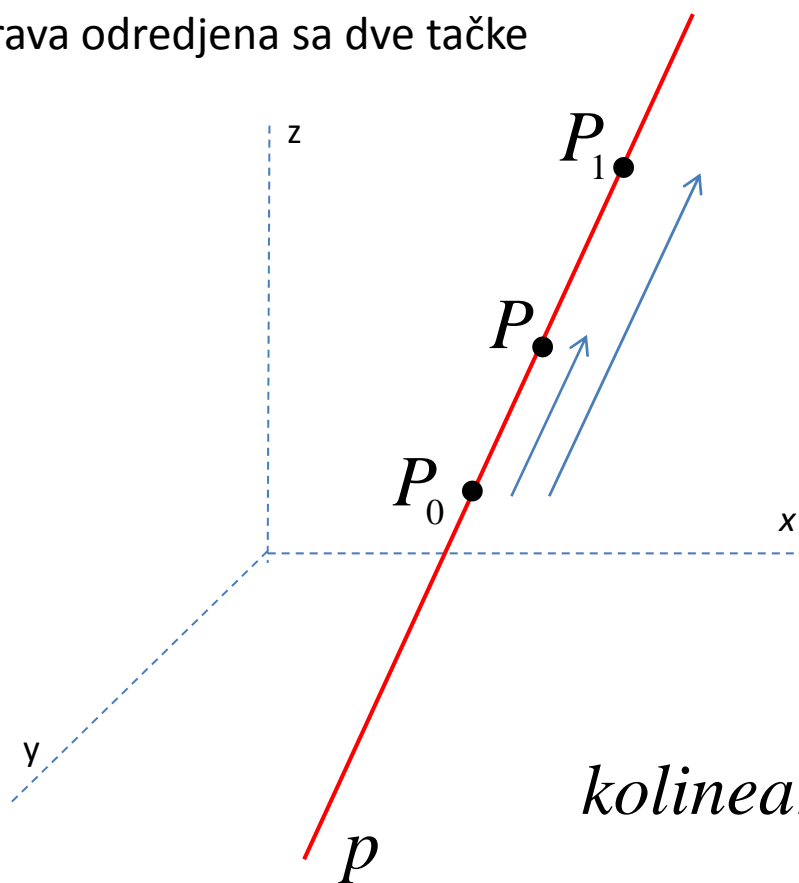
---

$$\frac{x - 1}{1} = \frac{y + 2}{3} = \frac{z - 3}{-1}$$

$$x - 1 = \frac{y + 2}{3} = -(z - 3)$$

## PARAMETARSKE JEDNAČINE PRAVE

Prava određena sa dve tačke



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

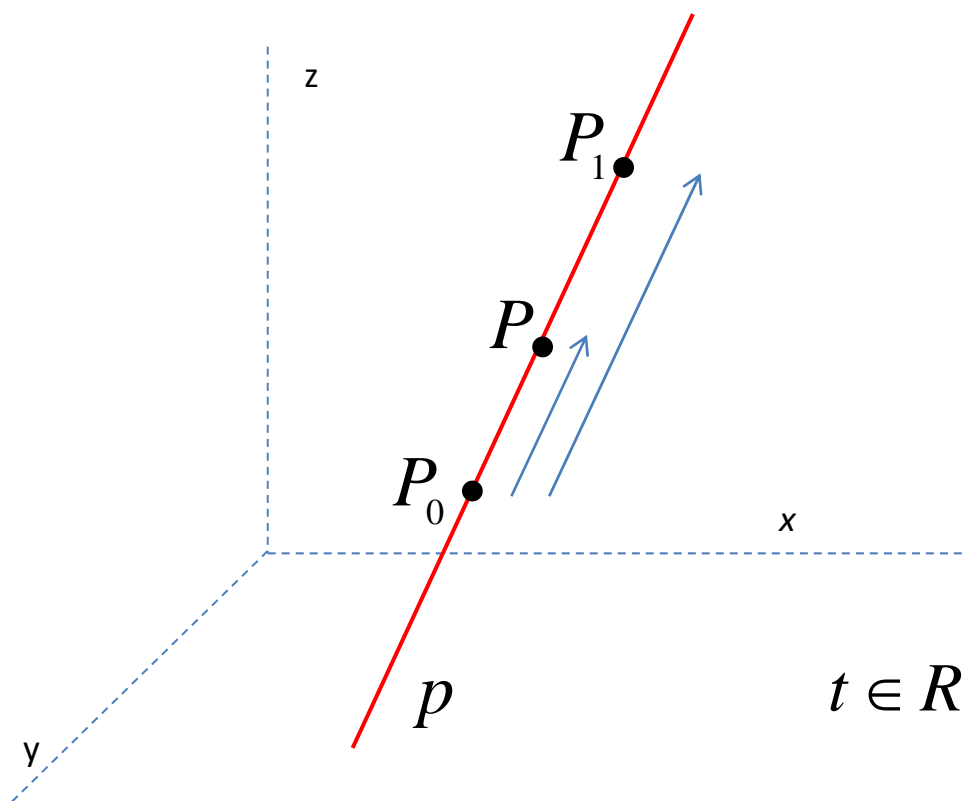
$$\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

$$\text{kolinearnost } \overrightarrow{P_0P} \text{ i } \vec{p} \Leftrightarrow \overrightarrow{P_0P} = t \overrightarrow{P_0P_1}$$

$$(x - x_0, y - y_0, z - z_0) = t(x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

$$x - x_0 = t(x_1 - x_0), \quad y - y_0 = t(y_1 - y_0), \quad z - z_0 = t(z_1 - z_0)$$

## PARAMETARSKE JEDNAČINE PRAVE



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

---


$$P(x, y, z) \in p$$

$$x - x_0 = t(x_1 - x_0)$$

$$y - y_0 = t(y_1 - y_0)$$

$$z - z_0 = t(z_1 - z_0)$$


---

$t \in \mathbb{R}$

$$x = x_0 + t(x_1 - x_0)$$

$$t \in \mathbb{R} \quad y = y_0 + t(y_1 - y_0)$$

$$z = z_0 + t(z_1 - z_0)$$

$t \in \mathbb{R}$

$$x = (1-t)x_0 + t x_1$$

$$y = (1-t)y_0 + t y_1$$

$$z = (1-t)z_0 + t z_1$$

PRIMER

Napisati parametarske jednačine prave određene sa dve tacke

$$P_0(1, -2, 3) \quad \text{i} \quad P_1(3, 4, 1) \quad .$$

$p:$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$x = x_0 + t(x_1 - x_0)$$

$$y = y_0 + t(y_1 - y_0)$$

$$z = z_0 + t(z_1 - z_0)$$

$$t \in R$$

$$p: P_0(1, -2, 3) \quad P_1(3, 4, 1)$$

---


$$x = 1 + t(3 - 1)$$

$$y = -2 + t(4 + 2) \quad t \in R$$

$$z = 3 + t(1 - 3)$$

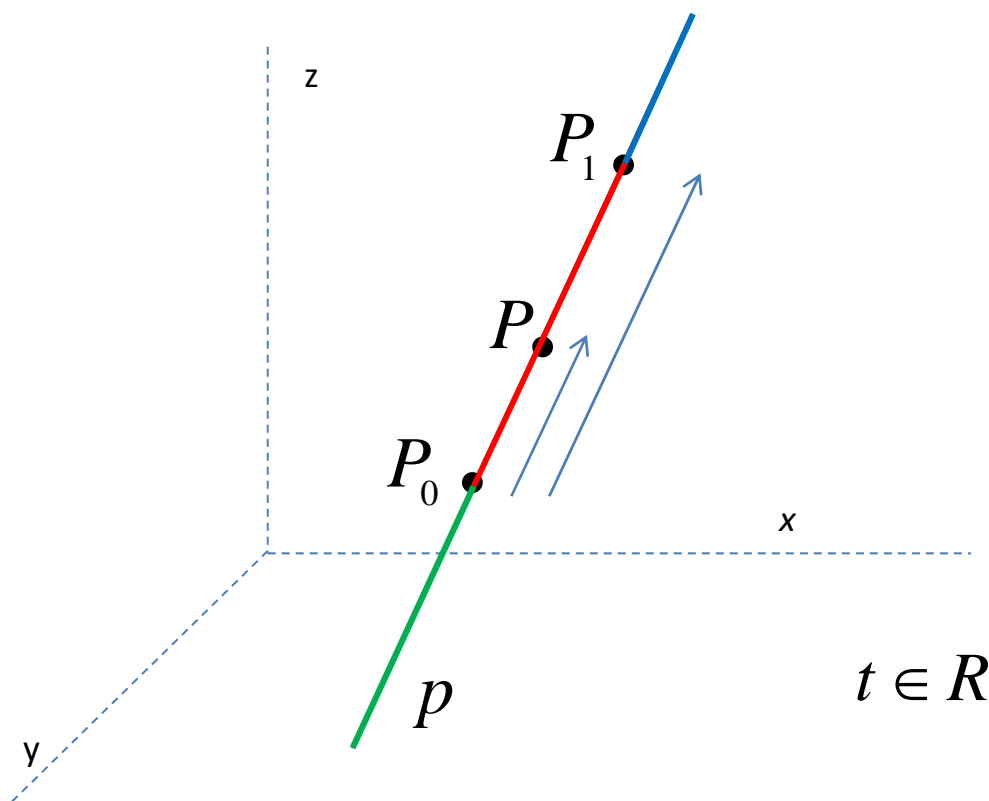
---


$$x = 1 + 2t$$

$$y = -2 + 6t \quad t \in R$$

$$z = 3 - 2t$$

# PARAMETARSKE JEDNAČINE PRAVE



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$x = x_0 + t(x_1 - x_0)$$

$$y = y_0 + t(y_1 - y_0)$$

$$z = z_0 + t(z_1 - z_0)$$

$$t \in \mathbb{R}$$

$$\overrightarrow{P_0P} = t \overrightarrow{P_0P_1}$$

$$t < 0$$

$$t = 0$$

$$0 < t < 1$$

$$t = 1$$

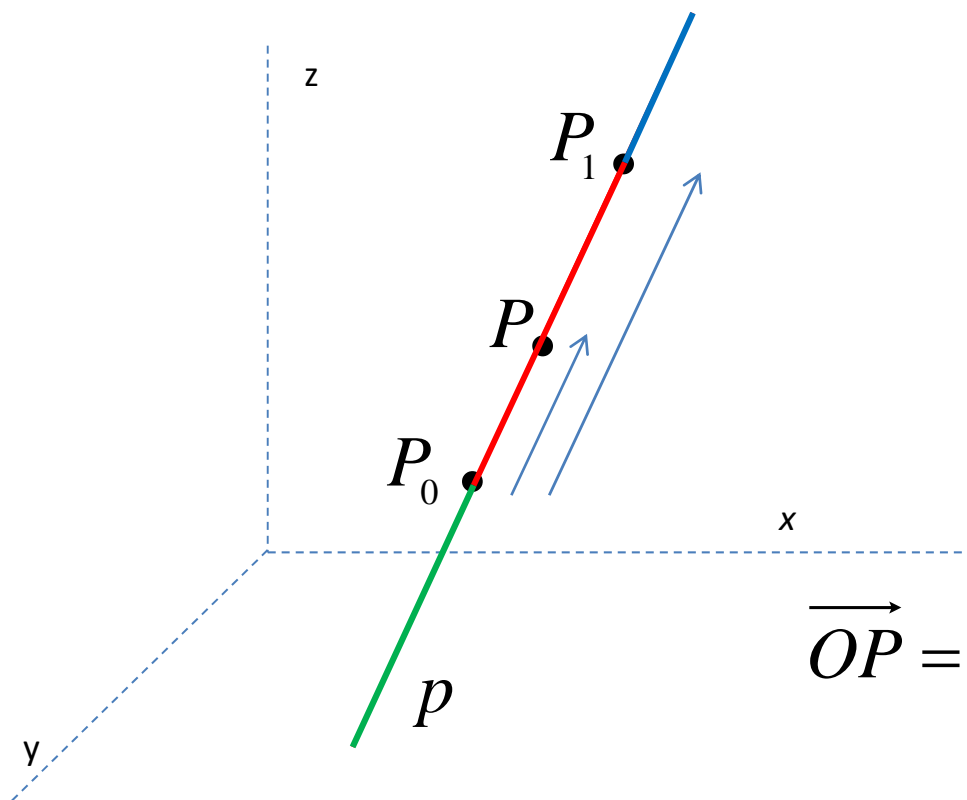
$$t > 1$$

$$P_0$$

$$P_1$$



# PARAMETARSKE JEDNAČINE PRAVE



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$\vec{OP} = \vec{OP}_0 + \vec{P}_0P = \vec{OP}_0 + t \vec{P}_0P_1$$

$$\vec{P}_0P = t \vec{P}_0P_1$$

$$t < 0$$

$$t = 0$$

$$0 < t < 1$$

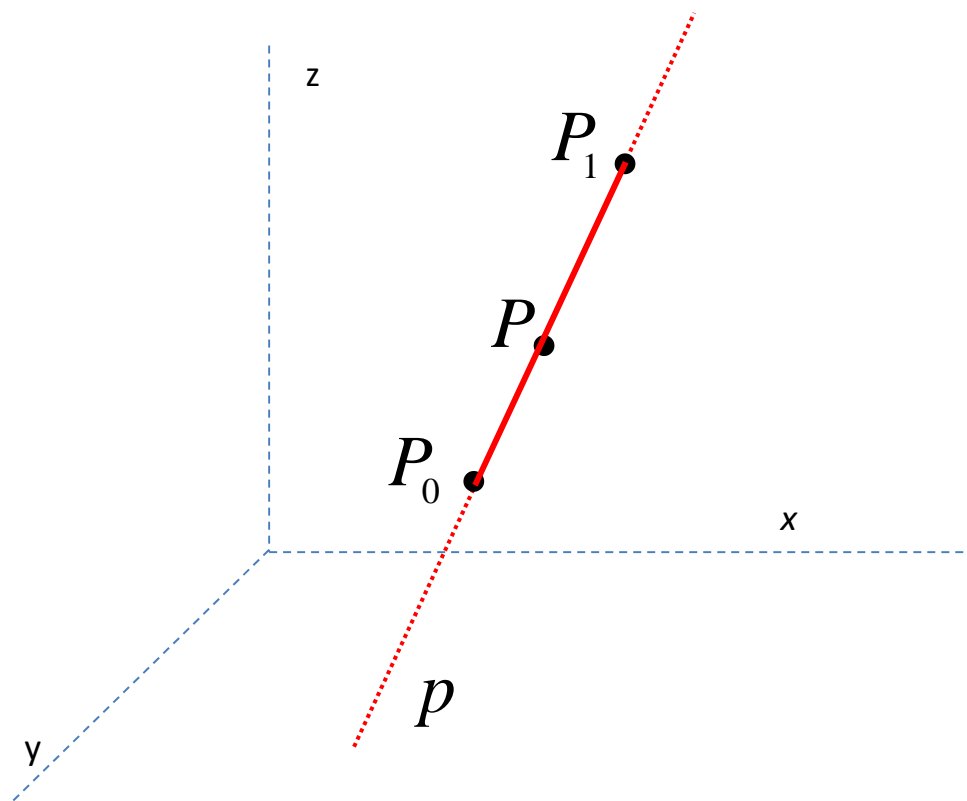
$$t = 1$$

$$t > 1$$

$P_0$

$P_1$

## PARAMETARSKE JEDNAČINE DUŽI



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

---

$$P(x, y, z) \in P_0P_1$$

Jednačine duži  $P_0P_1$

$$x = x_0 + t(x_1 - x_0)$$

$$y = y_0 + t(y_1 - y_0)$$

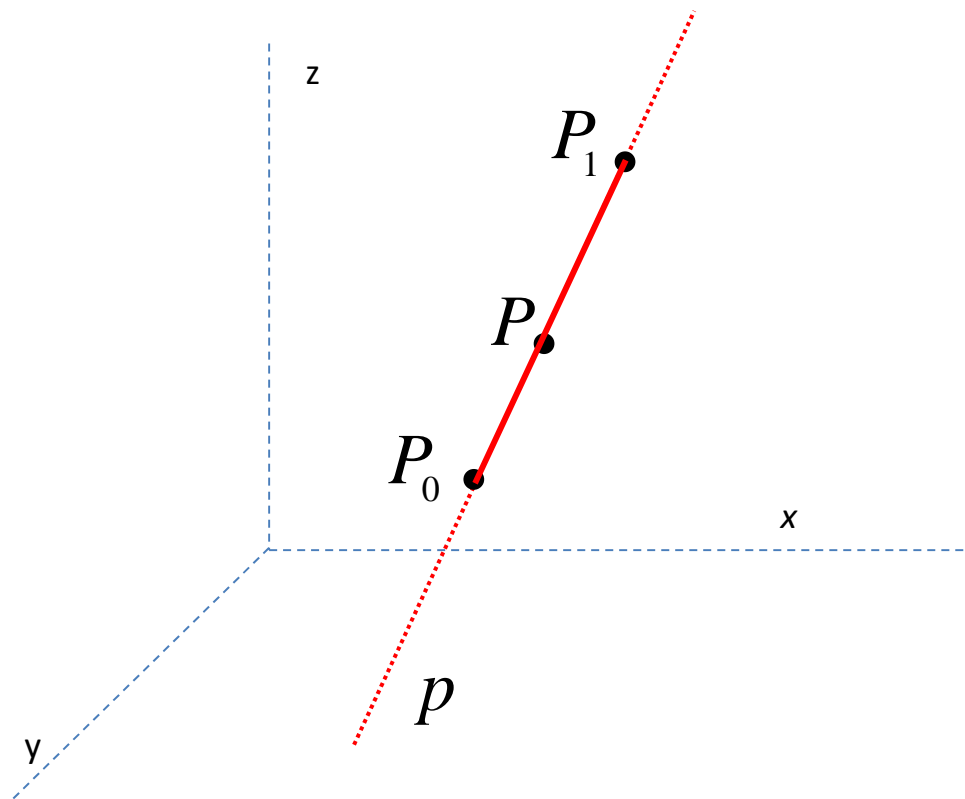
$$z = z_0 + t(z_1 - z_0)$$

---

$$0 \leq t \leq 1$$

---

## PARAMETARSKÉ JEDNAČINE DUŽI



$p$ :

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

---

$$P(x, y, z) \in P_0P_1$$

Jednačine duži  $P_0P_1$

$$x = (1-t)x_0 + tx_1$$

$$y = (1-t)y_0 + ty_1$$

$$z = (1-t)z_0 + tz_1$$

---

$$0 \leq t \leq 1$$

---

## PARAMETARSKE JEDNAČINE PRAVE

Prelazak sa kanonskog na parametarski oblik

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

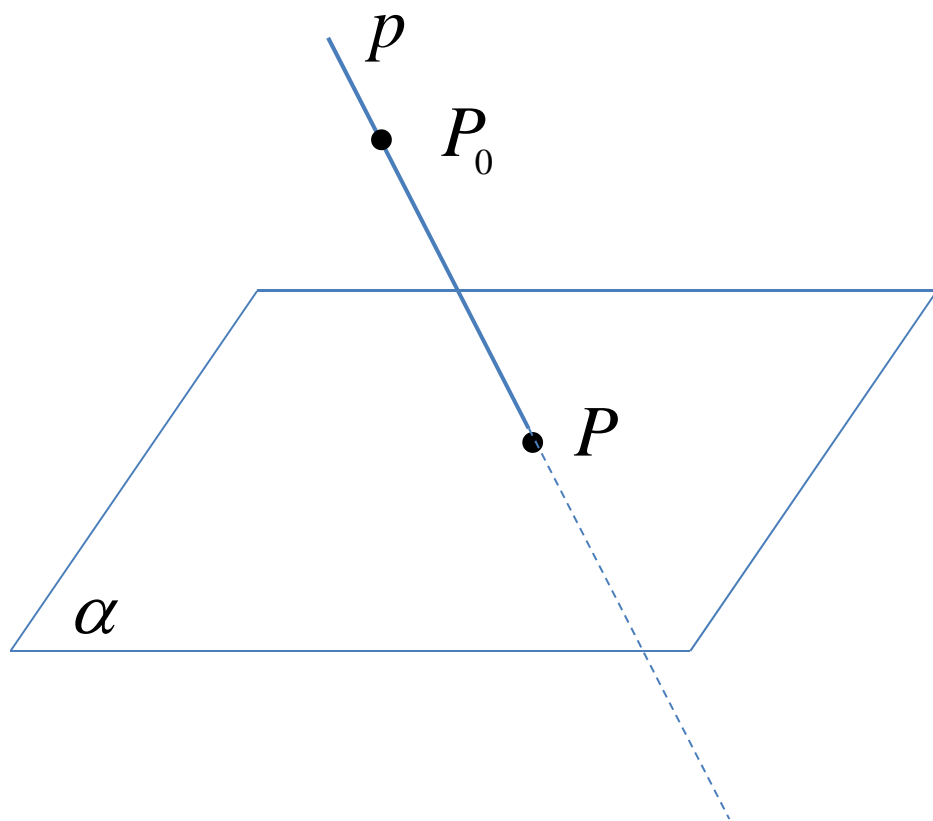
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t \quad t \in R$$

$$x = x_0 + at$$

$$y = y_0 + bt \quad t \in R$$

$$z = z_0 + ct$$

## PRODOR PRAVE KROZ RAVAN



$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\alpha: Ax + By + Cz + D = 0$$

---

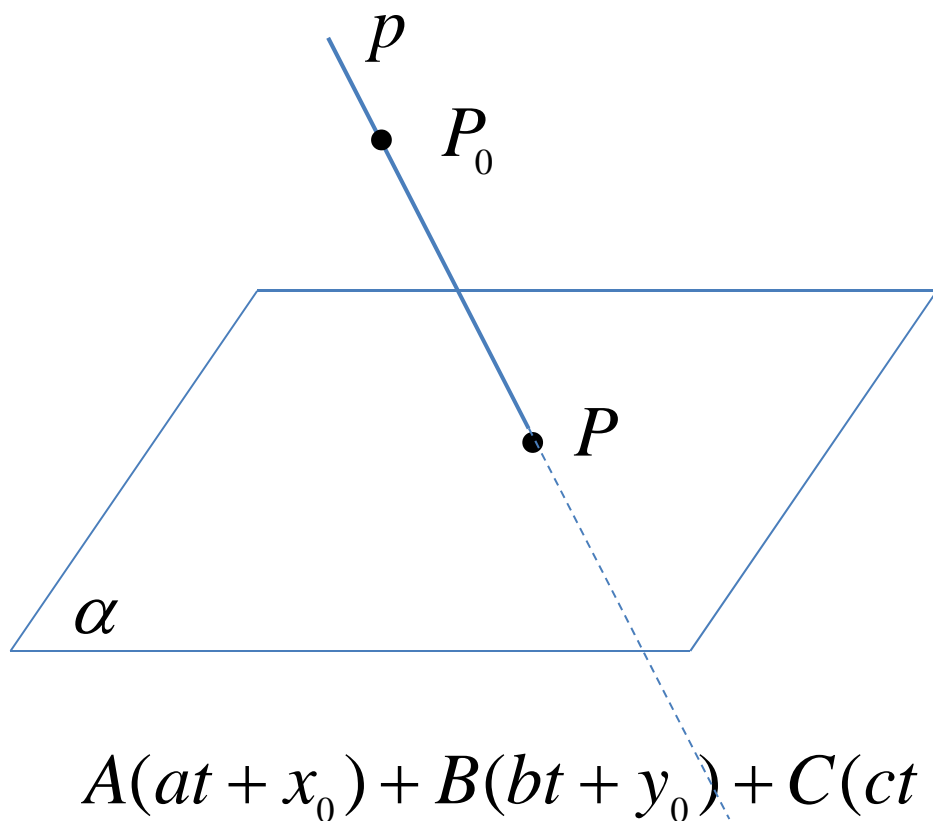
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

$$x = at + x_0$$

$$y = bt + y_0 \quad t \in \mathbb{R}$$

$$z = ct + z_0$$

PRODOR PRAVE KROZ RAVAN



$$P = p \cap \alpha$$

$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\alpha: Ax + By + Cz + D = 0$$

---


$$x = at + x_0 \quad t \in \mathbb{R}$$

$$y = bt + y_0$$

$$z = ct + z_0$$


---

$$P(x_P, y_P, z_P) = ?$$

$$A(at + x_0) + B(bt + y_0) + C(ct + z_0) + D = 0 \quad \Rightarrow \quad t = t_P$$

$$x_P = at_P + x_0, \quad y_P = bt_P + y_0, \quad z_P = ct_P + z_0 \quad \Rightarrow \quad P(x_P, y_P, z_P)$$

PRIMER

Odrediti prodor prave  $p: \frac{x+1}{-2} = \frac{y-2}{-3} = \frac{z+3}{2}$

kroz ravan  $\alpha: -2x - 3y + 2z + 27 = 0$

---

$$\frac{x+1}{-2} = \frac{y-2}{-3} = \frac{z+3}{2} = t \quad \begin{array}{l} \text{Parametrizacija} \\ \text{prave} \end{array}$$

$$x = -1 - 2t, \quad y = 2 - 3t, \quad z = -3 + 2t, \quad t \in \mathbb{R}$$

---

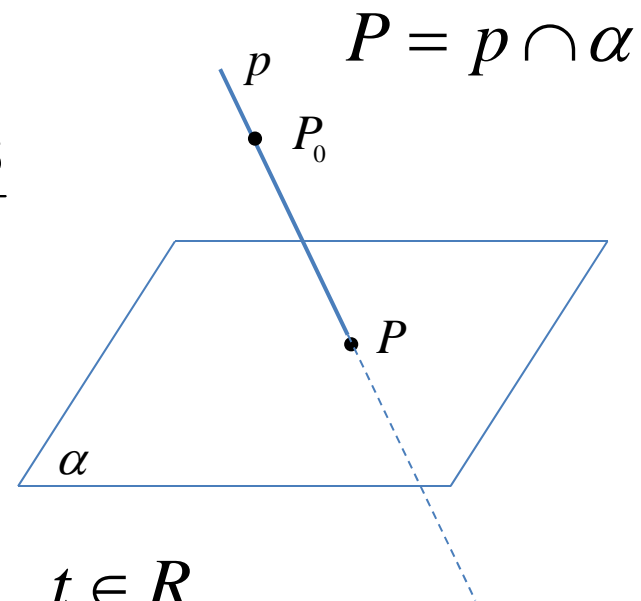
$$-2x - 3y + 2z + 27 = 0$$

$$-2(-1 - 2t) - 3(2 - 3t) + 2(-3 + 2t) + 27 = 0$$

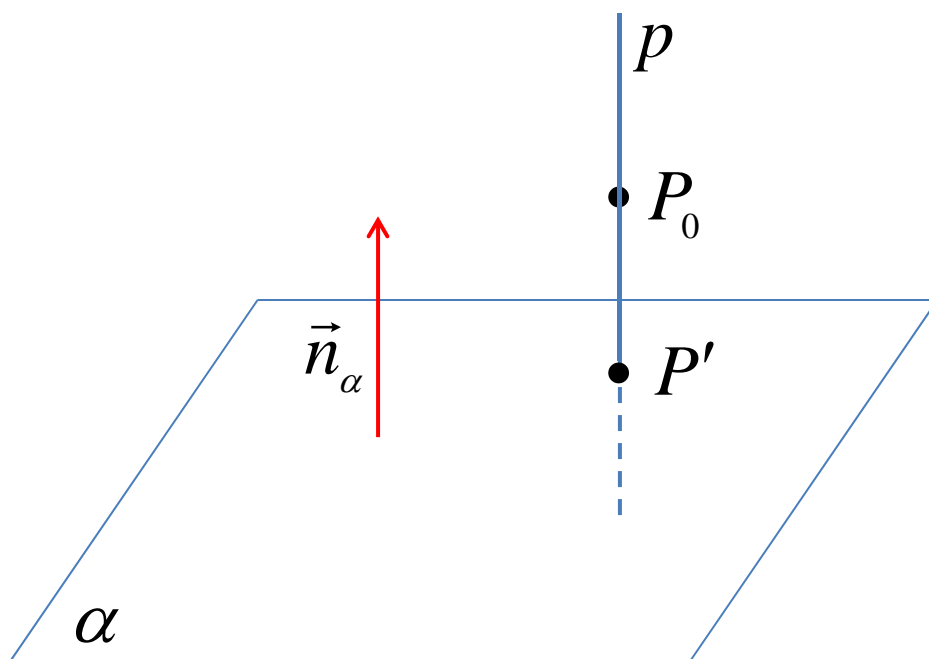
$$4t + 9t + 4t + 17 = 0 \quad 17t + 17 = 0 \quad t = -1$$

$$x_P = -1 - 2 \cdot (-1), \quad y_P = 2 - 3 \cdot (-1), \quad z_P = -3 + 2 \cdot (-1)$$

$$x_P = 1, \quad y_P = 5, \quad z_P = -5 \quad p \cap \alpha = P(1, 5, -5)$$



## PROJEKCIJA TAČKE NA RAVAN



$$P_0(x_0, y_0, z_0)$$

$$\alpha : Ax + By + Cz + D_1 = 0$$

$$\vec{n}_\alpha = (A, B, C)$$

---

$$p : (P_0 \in p, p \perp \alpha)$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = \vec{n}_\alpha$$

$$p : \frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

$$p \cap \alpha = P'$$



PRIMER

Odrediti ortogonalnu projekciju tačke

$$P_0(1, -2, 5)$$

na ravan

$$\alpha: 2x - 3y + z + 1 = 0$$

$$P_0 \in p, \quad p \perp \alpha$$

$$p: P_0(1, -2, 5)$$

$$\vec{p} = \vec{n}_\alpha = (2, -3, 1)$$

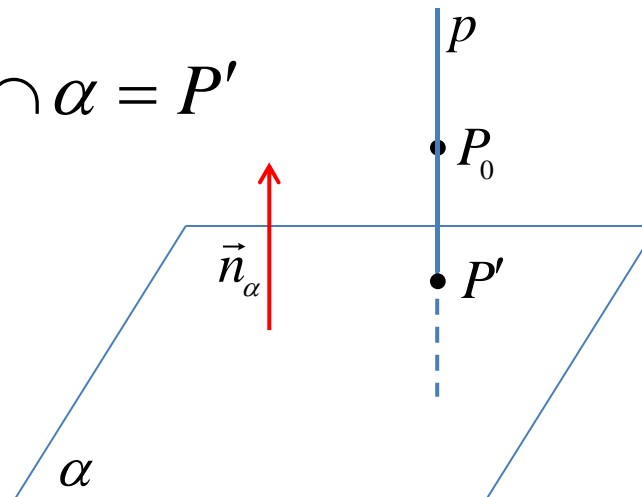
$$p: \frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-5}{1} = t$$

$$x = 2t + 1$$

$$y = -3t - 2 \quad t \in \mathbb{R}$$

$$z = t + 5$$

$$p \cap \alpha = P'$$



$$2(2t + 1) - 3(-3t - 2) + (t + 5) + 1 = 0$$

$$14t + 14 = 0 \quad t = -1$$

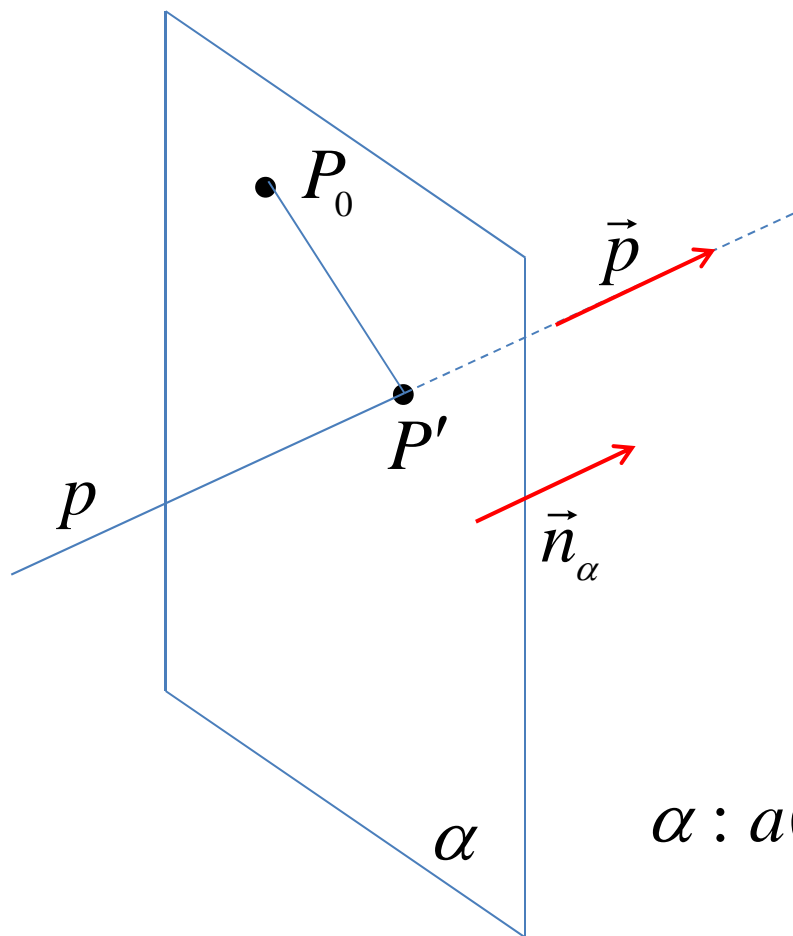
$$x_{P'} = 2 \cdot (-1) + 1 = -1$$

$$y_{P'} = -3 \cdot (-1) - 2 = 1$$

$$z_{P'} = -1 + 5 = 4$$

$$p \cap \alpha = P'(-1, 1, 4)$$

## PROJEKCIJA TAČKE NA PRAVU



$$P_0(x_0, y_0, z_0)$$

$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\vec{p} = (a, b, c)$$

---


$$\alpha: (P_0 \in \alpha, \alpha \perp p)$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{n}_\alpha = \vec{p}$$

---


$$\vec{n}_\alpha = (a, b, c)$$

$$\alpha: a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$p \cap \alpha = P'$$

PRIMER

Odrediti ortogonalnu projekciju tačke  $P_0(2, 2, 1)$

na pravu  $p: \frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-5}{1}$

---


$$\alpha: (P_0 \in \alpha, \alpha \perp p)$$

$$P_0(2, 2, 1)$$

$$\vec{n}_\alpha = \vec{p} = (2, -3, 1)$$

---


$$\alpha: 2(x-2) - 3(y-2) + (z-1) = 0$$

$$2x - 3y + z + 1 = 0$$

---

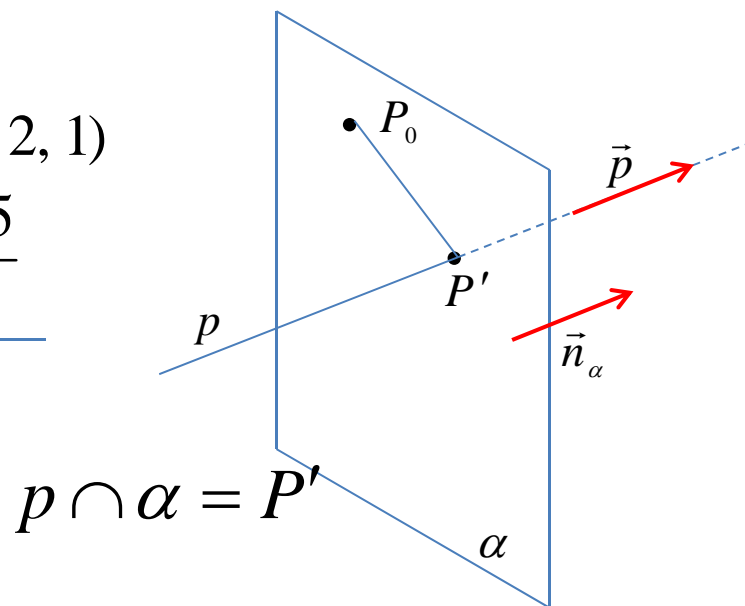

$$p: x = 2t + 1, y = -3t - 2, z = t + 5$$

$$2(2t + 1) - 3(-3t - 2) + (t + 5) + 1 = 0$$

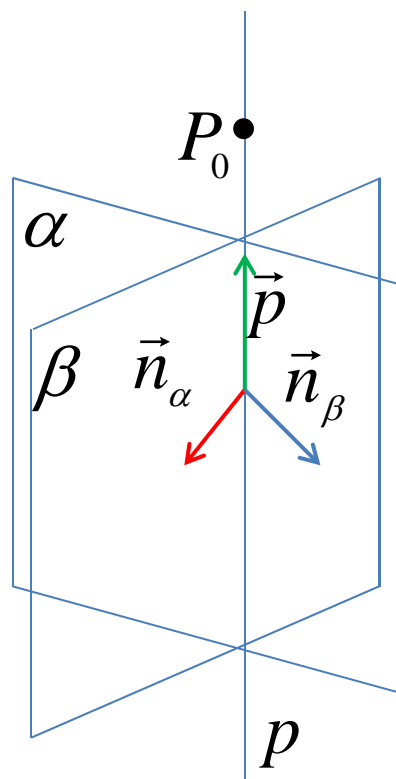
$$14t + 14 = 0 \Rightarrow t = -1$$

$$x_{P'} = -1, y_{P'} = 1, z_{P'} = 4$$

$$p \cap \alpha = P'(-1, 1, 4)$$



## PRAVA KAO PRESEK DVE RAVNI



$$p:$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (a, b, c)$$

$$P_0(x_0, y_0, z_0) \in p$$

$$\alpha: A_1x + B_1y + C_1z + D_1 = 0$$

$$\beta: A_2x + B_2y + C_2z + D_2 = 0$$

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$$p = \alpha \cap \beta$$


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$$\vec{n}_\alpha = (A_1, B_1, C_1) \quad \vec{n}_\beta = (A_2, B_2, C_2)$$

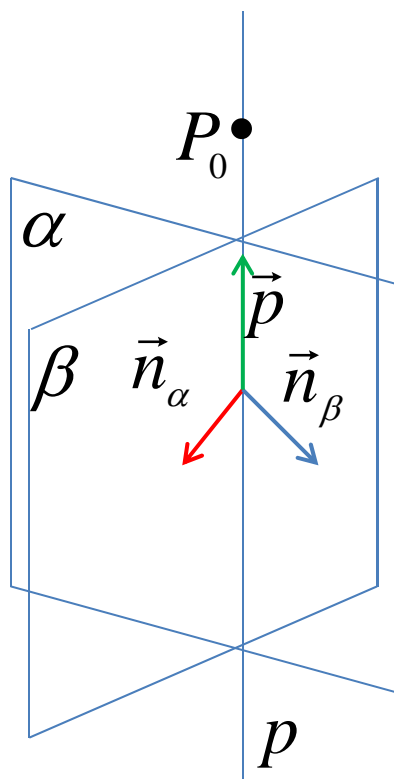
$$\vec{n}_\alpha \perp \alpha \Rightarrow \vec{n}_\alpha \perp p$$

$$\vec{n}_\beta \perp \beta \Rightarrow \vec{n}_\beta \perp p$$

$$\vec{p} \perp \vec{n}_\alpha, \vec{p} \perp \vec{n}_\beta \Rightarrow \vec{p} = \vec{n}_\alpha \times \vec{n}_\beta$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

## PRAVA KAO PRESEK DVE RAVNI



$$\alpha: A_1x + B_1y + C_1z + D_1 = 0$$

$$\beta: A_2x + B_2y + C_2z + D_2 = 0$$

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$$p = \alpha \cap \beta$$

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$$p: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$\vec{n}_\alpha = (A_1, B_1, C_1) \quad \vec{n}_\beta = (A_2, B_2, C_2)$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (a, b, c)$$

Kanonski oblik:

$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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Napisati kanonske jednačine prave

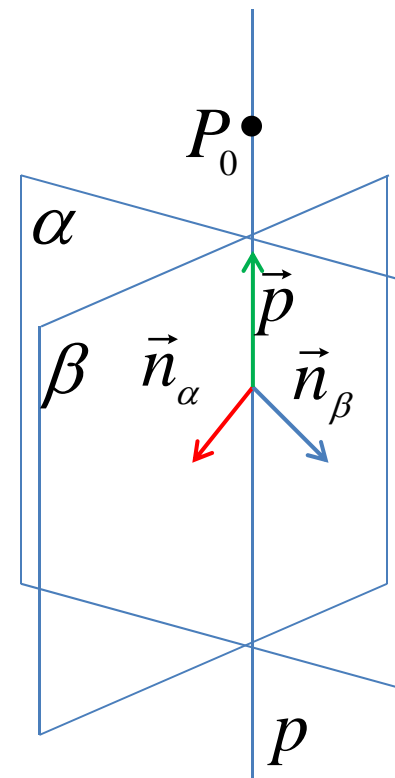
$$p: \begin{cases} 2x - 3y + z + 1 = 0 \\ -x + y - 3 = 0 \end{cases}$$

zadate kao presek dve ravni.

$$\vec{n}_\alpha = (2, -3, 1) \quad \vec{n}_\beta = (-1, 1, 0)$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} -3 & 1 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} = -\vec{i} - \vec{j} - \vec{k}$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (-1, -1, -1)$$



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Napisati kanonske jednačine prave

$$p: \begin{cases} 2x - 3y + z + 1 = 0 \\ -x + y - 3 = 0 \end{cases}$$

zadate kao presek dve ravni.

$$\vec{n}_\alpha = (2, -3, 1) \quad \vec{n}_\beta = (-1, 1, 0)$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (-1, -1, -1)$$

$$P_0(x_0, y_0, z_0) \in p$$

$$2x - 3y + z + 1 = 0$$

$$-x + y - 3 = 0$$

$$z = 0$$

$$2x - 3y + 1 = 0$$

$$-x + y - 3 = 0 \quad | \cdot 2$$

$$-y - 5 = 0 \Rightarrow y = -5$$

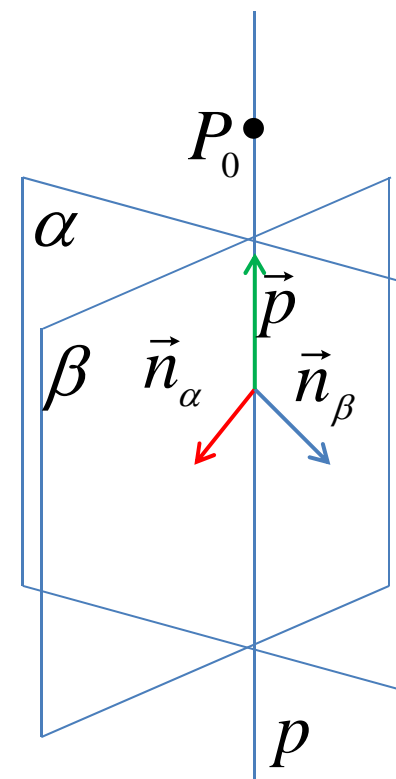
$$-x - 5 - 3 = 0 \Rightarrow x = 8$$

$$p: P_0(8, -5, 0)$$

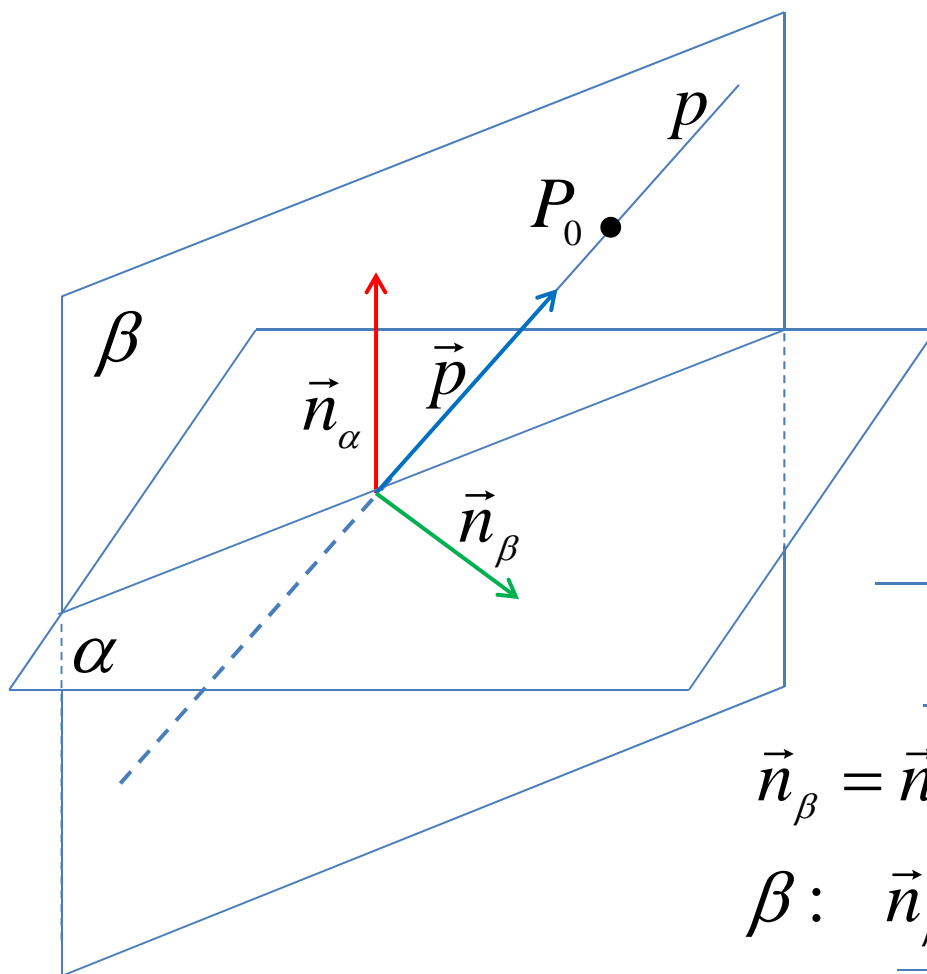
$$\vec{p} = (-1, -1, -1)$$

$$p: \frac{x-8}{-1} = \frac{y+5}{-1} = \frac{z}{-1} \quad | \cdot (-1)$$

$$p: x - 2 = y - 5 = z$$



## PROJEKCIJA PRAVE NA RAVAN



$$\alpha: A_1x + B_1y + C_1z + D_1 = 0$$

$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\vec{n}_\alpha = (A_1, B_1, C_1)$$

$$\vec{p} = (a, b, c)$$

$$\beta \supset p, \quad \beta \perp \alpha$$

$$\vec{n}_\beta = \vec{n}_\alpha \times \vec{p}, \quad P_0(x_0, y_0, z_0) \in p \subset \beta$$

$$\beta: \vec{n}_\beta = \vec{n}_\alpha \times \vec{p}, \quad P_0(x_0, y_0, z_0) \in \beta$$

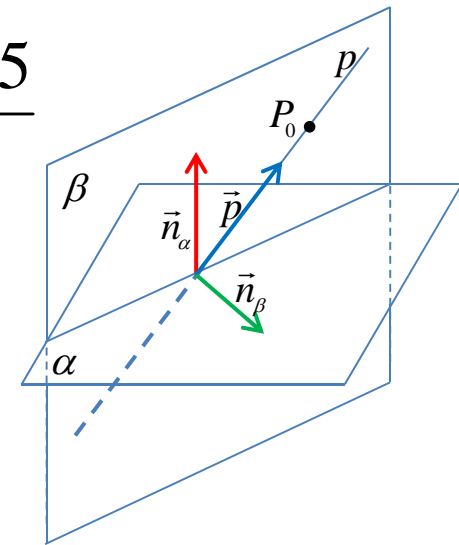
$$\alpha \cap \beta = p'$$



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Odrediti projekciju prave  $p: \frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-5}{1}$

na ravan  $\alpha: x - 2z + 4 = 0$ .



$$\vec{p} = (2, -3, 1) \quad \vec{n}_\alpha = (1, 0, -2)$$

$$\beta: \vec{n}_\beta = \vec{n}_\alpha \times \vec{p} \quad P_0(1, -2, 5) \in p \subset \beta$$

$$\vec{n}_\beta = \vec{n}_\alpha \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 0 & -2 \end{vmatrix} = \vec{i} \begin{vmatrix} -3 & 1 \\ 0 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 6\vec{i} + 5\vec{j} + 3\vec{k}$$

$$\beta: 6(x-1) + 5(y+2) + 3(z-5) = 0$$

$$\beta: 6x + 5y + 3z - 11 = 0$$

$$p' = \alpha \cap \beta$$

$$p': \begin{cases} x - 2z + 4 = 0 \\ 6x + 5y + 3z - 11 = 0 \end{cases}$$

## UGAO IZMEDJU DVE PRAVE

$$\vec{p}_1 \cdot \vec{p}_2 = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

Ugao izmedju dve prave je ugao za koji treba zarotirati jednu pravu da bi se poklopila sa drugom pravom.

$$p_1 : \frac{x - x_0}{a_1} = \frac{y - y_0}{b_1} = \frac{z - z_0}{c_1}$$

$$p_2 : \frac{x - x_0}{a_2} = \frac{y - y_0}{b_2} = \frac{z - z_0}{c_2}$$

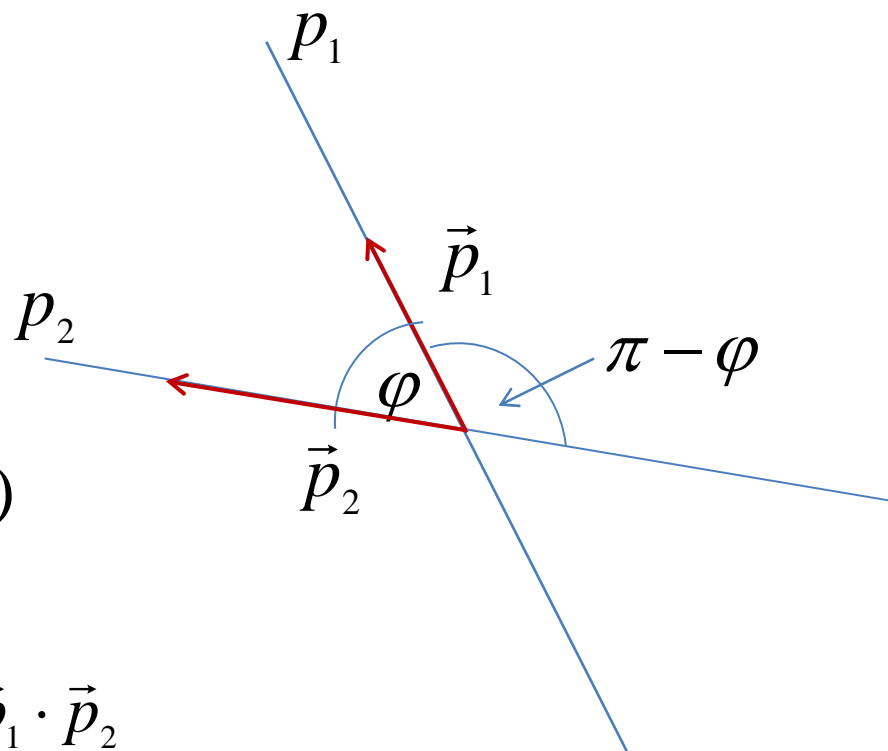
$$\vec{p}_1 = (a_1, b_1, c_1) \quad \vec{p}_2 = (a_2, b_2, c_2)$$

$$\vec{p}_1 \cdot \vec{p}_2 = |\vec{p}_1| \cdot |\vec{p}_2| \cdot \cos \varphi$$

$$\cos \varphi = \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| \cdot |\vec{p}_2|} \quad \varphi = \arccos \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| \cdot |\vec{p}_2|}$$

$$\varphi, \quad \pi - \varphi$$

su dva ugla od kojih je jedan ostar a drugi tup



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Odrediti ugao između pravih

$$p_1: \frac{x-2}{1} = \frac{y-3}{3} = \frac{z+1}{-2}$$

$$p_2: x-4 = -(x+1) = -z$$

$$\vec{p}_1 = (1, 3, -2)$$

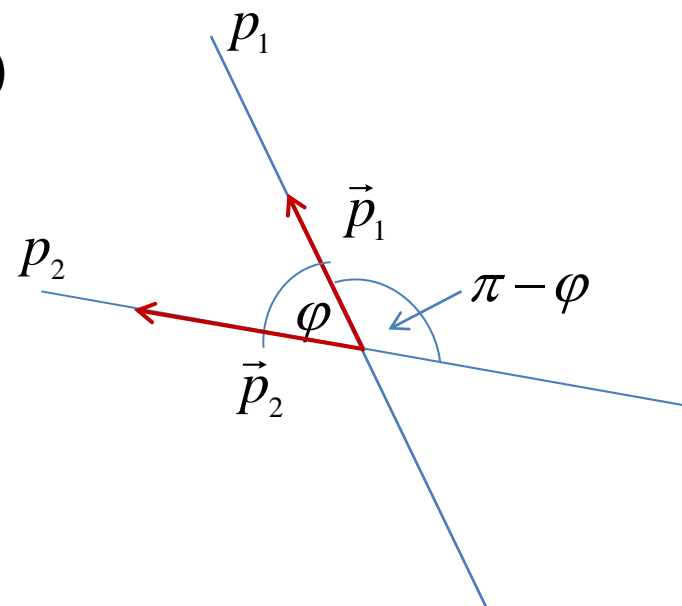
$$\vec{p}_2 = (1, -1, -1)$$

$$\vec{p}_1 \cdot \vec{p}_2 = 1 \cdot 1 + 3 \cdot (-1) + (-2) \cdot (-1) = 0$$

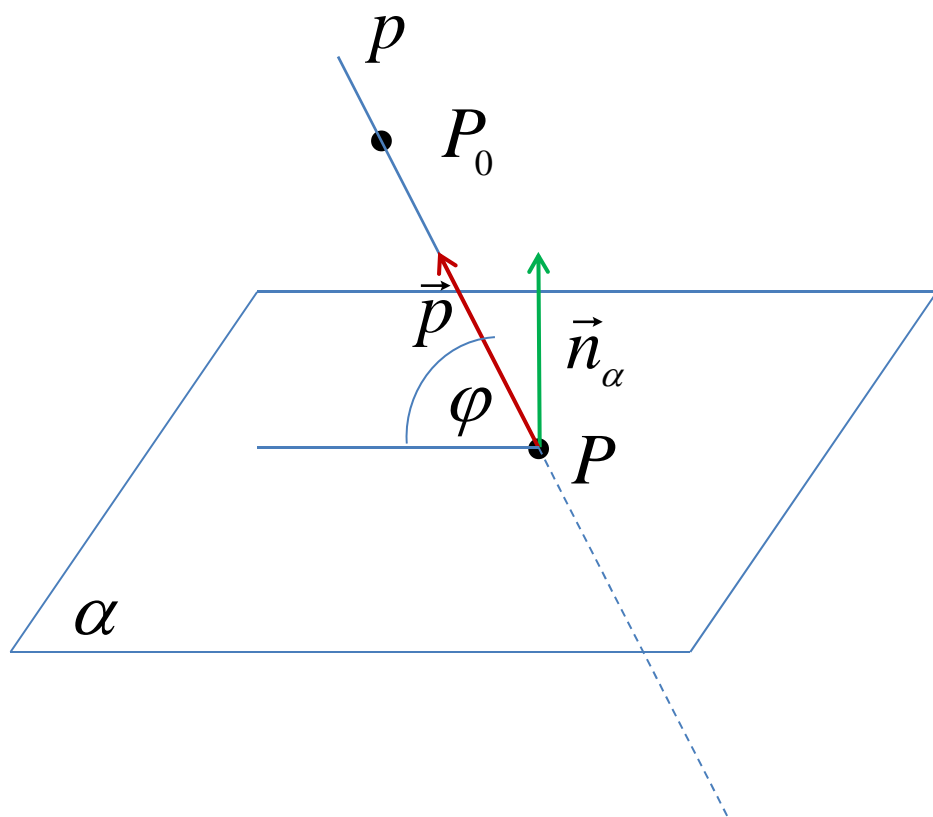
$$\cos \varphi = \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| \cdot |\vec{p}_2|} = \frac{0}{|\vec{p}_1| \cdot |\vec{p}_2|} = 0$$

$$\varphi = \arccos 0 = \frac{\pi}{2}$$

$$\pi - \varphi = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$



## UGAO IZMEDJU PRAVE I RAVNI



$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\alpha: Ax + By + Cz + D = 0$$

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$$\vec{p} = (a, b, c)$$

$$\vec{n}_\alpha = (A, B, C)$$

$$\angle(\vec{p}, \vec{n}_\alpha) = \frac{\pi}{2} - \varphi$$

$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{\vec{p} \cdot \vec{n}}{|\vec{p}| |\vec{n}_\alpha|}$$

PRIMER

Odrediti ugao izmedju prave

$$p: \frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{2}$$

$$\vec{p} = (1, 2, 2)$$

i ravni  $\alpha: 2x - y + 1 = 0$

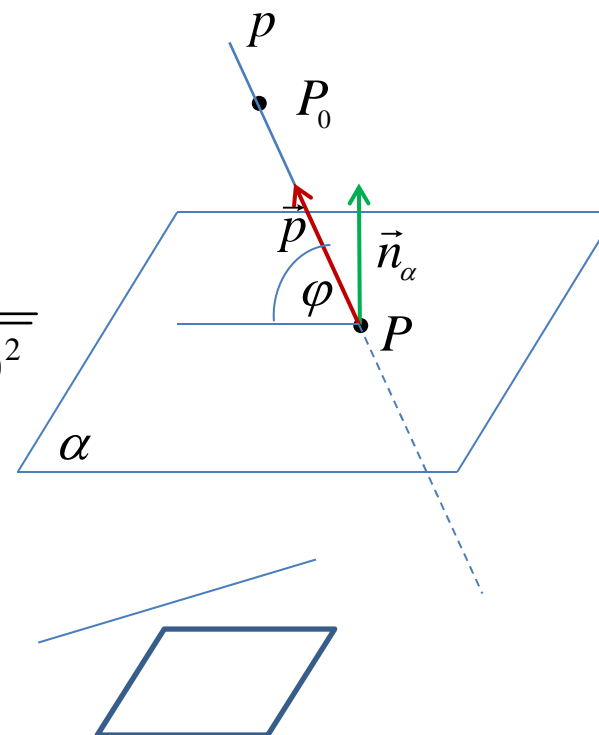
$$\vec{n}_\alpha = (2, -1, 0)$$

$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{\vec{p} \cdot \vec{n}}{|\vec{p}| |\vec{n}_\alpha|}$$

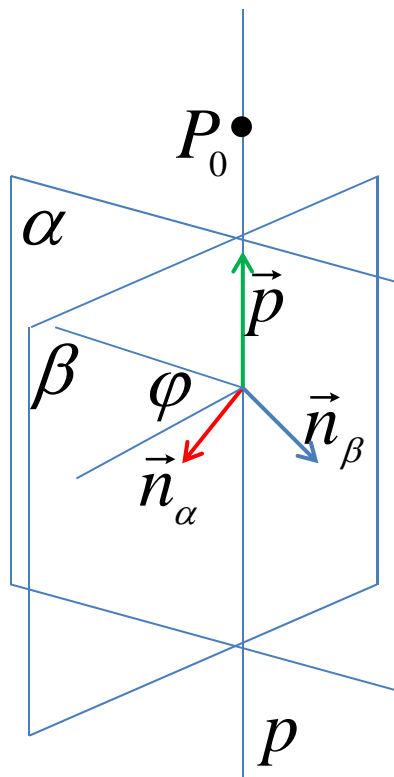
$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{1 \cdot 2 + 2 \cdot (-1) + 2 \cdot 0}{\sqrt{1^2 + 2^2 + 2^2} \cdot \sqrt{2^2 + (-1)^2 + 0^2}}$$

$$\cos\left(\frac{\pi}{2} - \varphi\right) = 0$$

$$\frac{\pi}{2} - \varphi = \frac{\pi}{2} \Rightarrow \varphi = 0 \Rightarrow p \parallel \alpha$$



## UGAO IZMEDJU DVE RAVNI



Ugao izmedju dve ravni

$$\alpha: A_1x + B_1y + C_1z + D_1 = 0$$

$$\beta: A_2x + B_2y + C_2z + D_2 = 0$$

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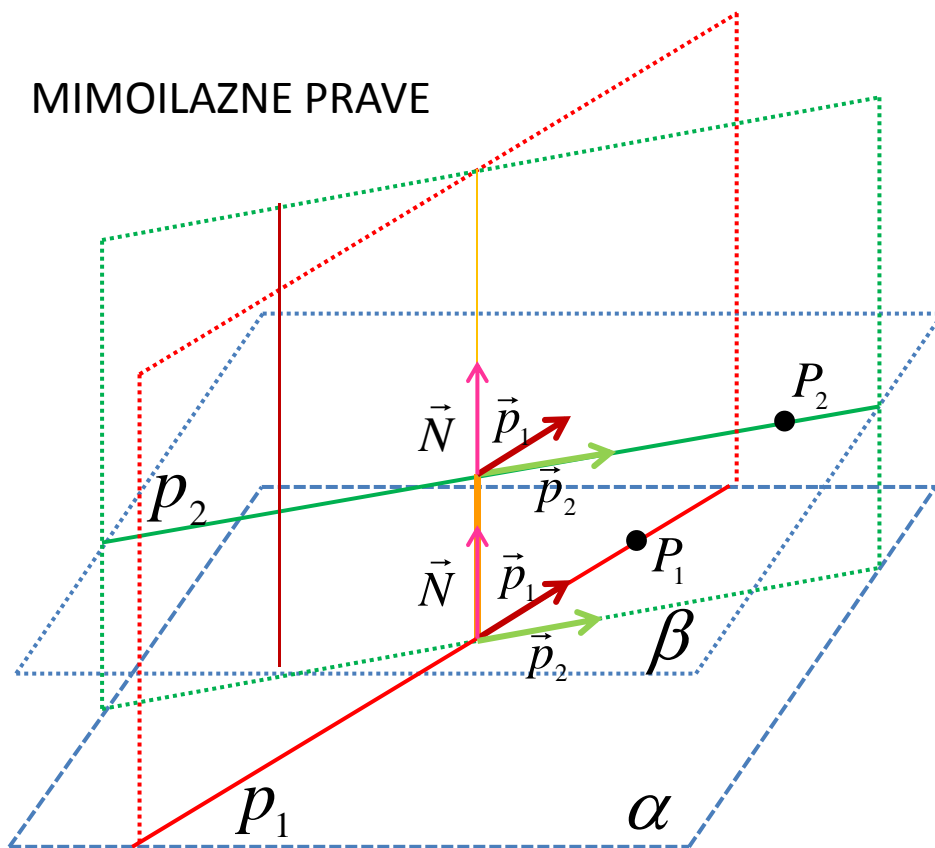
$$\vec{n}_\alpha = (A_1, B_1, C_1)$$

$$\vec{n}_\beta = (A_2, B_2, C_2)$$

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$$\cos \varphi = \frac{\vec{n}_\alpha \cdot \vec{n}_\beta}{|\vec{n}_\alpha| |\vec{n}_\beta|}$$

## MIMOILAZNE PRAVE



$$p_1: \frac{x - x_0}{a_1} = \frac{y - y_0}{b_1} = \frac{z - z_0}{c_1}$$

$$p_2: \frac{x - x_0}{a_2} = \frac{y - y_0}{b_2} = \frac{z - z_0}{c_2}$$

$$\vec{p}_1 = (a_1, b_1, c_1)$$

$$\vec{p}_2 = (a_2, b_2, c_2)$$

$\alpha$  Sadrži  $p_1$  i paralelna je  $p_2$ .

$\beta$  Sadrži  $p_2$  i paralelna je  $p_1$ .

$$\vec{N} = \vec{p}_1 \times \vec{p}_2 = (A, B, C)$$

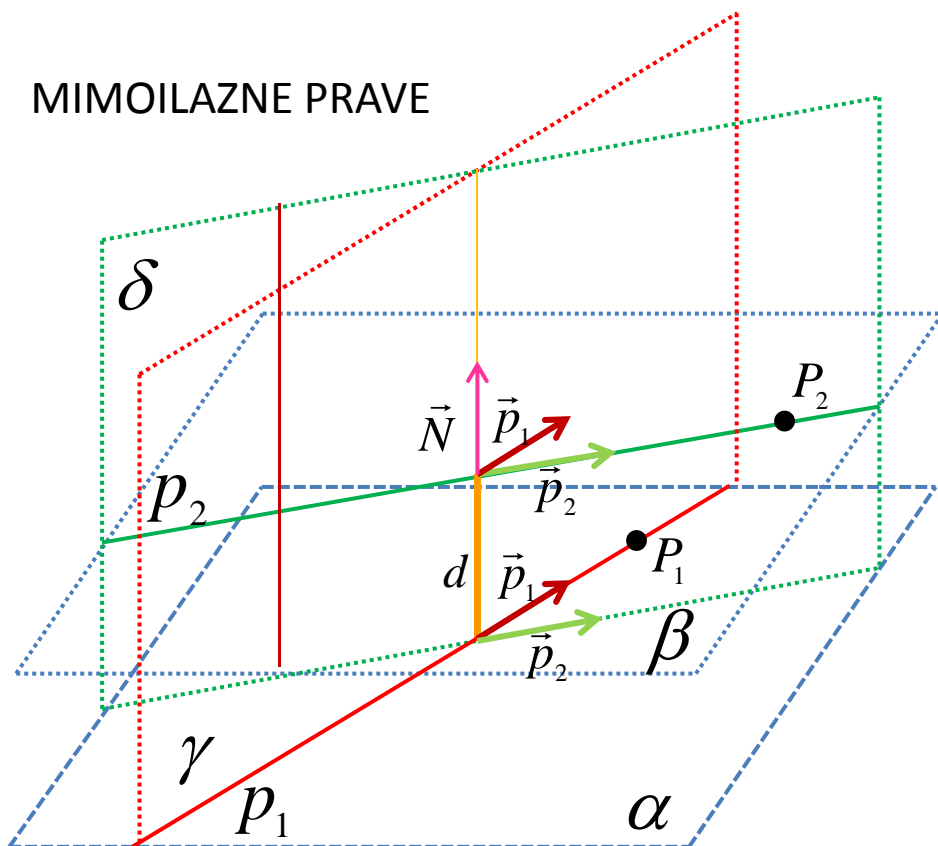
$$\alpha: \vec{N}, P_1 \in p_1$$

$$\alpha: Ax + By + Cz + D_1 = 0$$

$$\beta: \vec{N}, P_2 \in p_2$$

$$\beta: Ax + By + Cz + D_2 = 0$$

## MIMOILAZNE PRAVE



Najkraće rastojanje izmedju tačaka mimoilazne prave

$$d = \left| \overrightarrow{P_1 P_2} \cdot \text{ort } \vec{N} \right|$$

$$\gamma: \gamma \supseteq p_1, \gamma \perp \alpha \quad (\gamma \perp \beta)$$

$$\delta: \delta \supseteq p_2, \delta \perp \alpha \quad (\delta \perp \beta)$$

$$\gamma: P_1 \in p_1, \vec{n}_\gamma = \vec{p}_1 \times \vec{N}$$

$$\delta: P_2 \in p_2, \vec{n}_\delta = \vec{p}_2 \times \vec{N}$$

$$n = \gamma \cap \delta \quad \text{Zajednička normala}$$

$$n \cap \alpha = M_1 \in p_1 \quad n \cap \beta = M_2 \in p_2 \quad \text{najbliže tačke}$$

Rastojanje izmedju tačaka  $M_1 \in p_1$  i  $M_2 \in p_2$  je najkraće rastojanje izmedju dve mimoilazne prave.



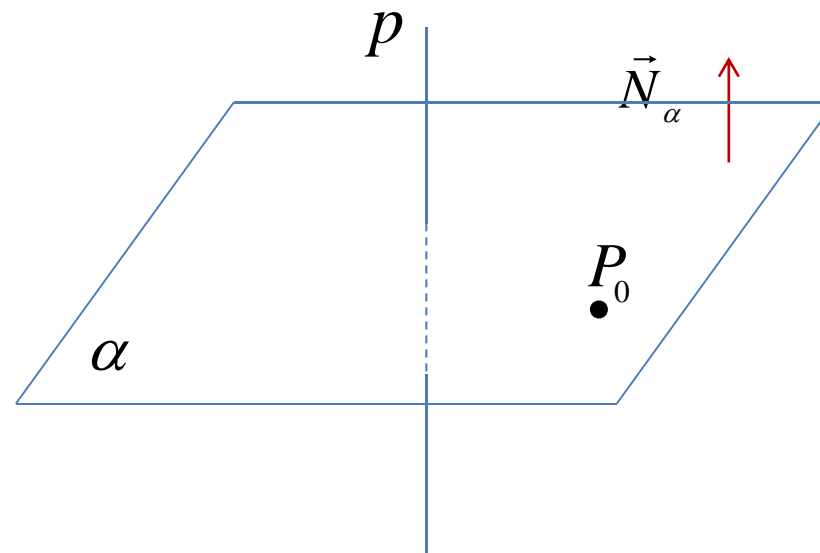
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Napisati jednačinu ravni  $\alpha$   
koja sadrži tačku

$$P_0(-2, 5, 3)$$

i normalna je na pravu

$$p: \frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{1}.$$



$$P_0(-2, 5, 3) \in \alpha \quad \vec{N}_\alpha = \vec{p} = (1, -1, 1)$$

$$\alpha: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$(x + 2) - (y - 5) + (z - 3) = 0$$

$$x - y + z + 4 = 0$$

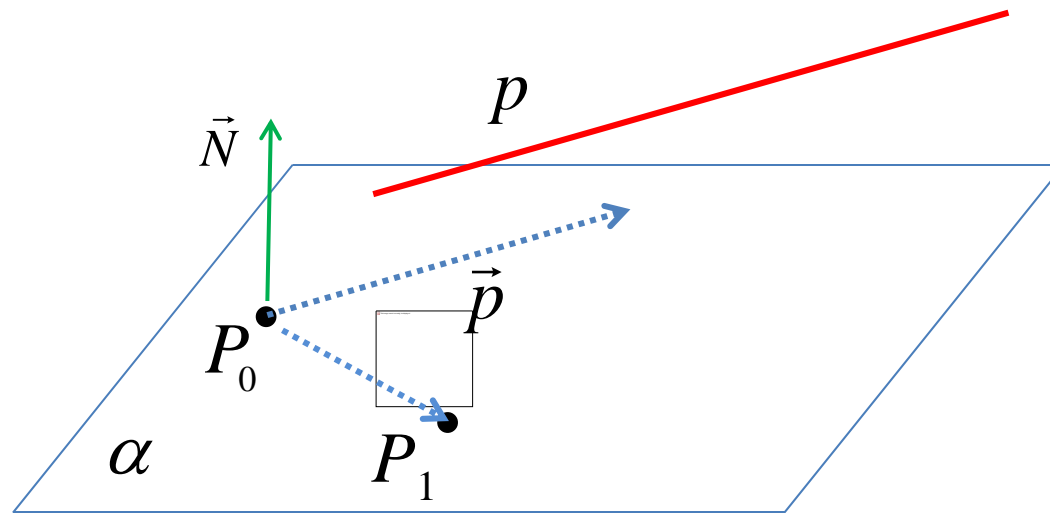
PRIMER

Napisati jednačinu ravni  $\alpha$   
koja sadrži tačke

$$P_0(-2, 1, 3) \quad \text{i} \quad P_1(0, 1, -3)$$

i paralelna je pravoj

$$\vec{p} : \frac{x-1}{1} = \frac{y+1}{2} = -\frac{z}{1}$$



$$\vec{a} = \overrightarrow{P_0P_1} = (0 + 2, 1 - 1, -3 - 3) = (2, 0, -6)$$

$$\vec{b} = \vec{p} = (1, 2, -1)$$

$$\vec{N} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -6 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & -6 \\ 2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -6 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 12\vec{i} - 4\vec{j} + 4\vec{k}$$

$$\vec{N} = (12, -4, 4) = (3, -1, 1)$$

$$3(x - 0) - (y - 1) + (z + 3) = 0$$

$$3x - y + z + 4 = 0$$

PRIMER

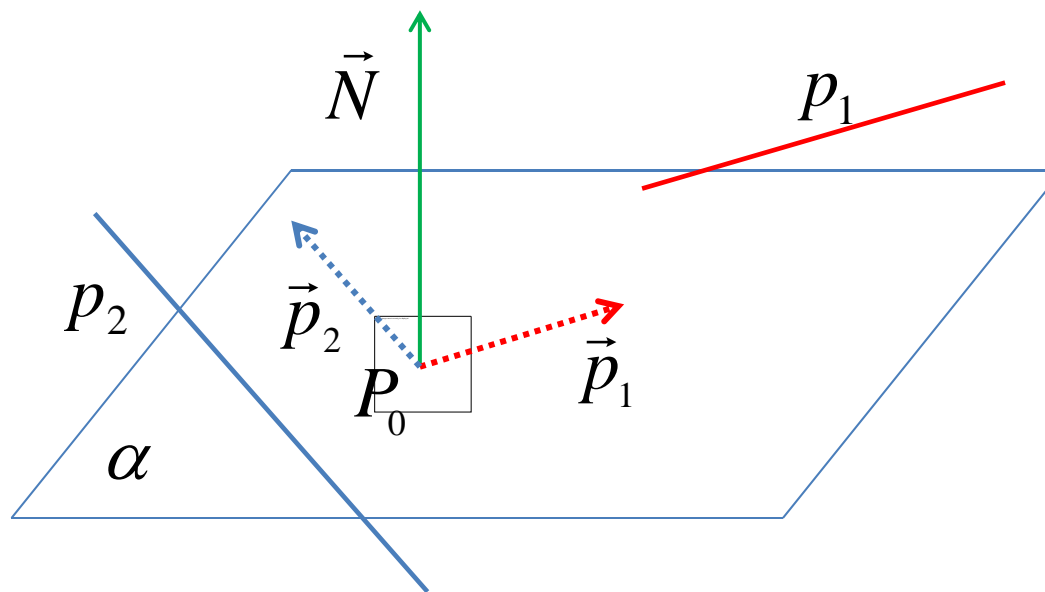
Napisati jednačinu ravni  $\alpha$   
koja sadrži tačku

$$P_0(2, 3, 0)$$

i paralelna je pravama

$$p_1: \frac{x+5}{1} = -\frac{y-5}{3} = -\frac{z-1}{2} \quad \text{i}$$

$$p_2: x = y = z$$



$$\vec{N} = \vec{p}_1 \times \vec{p}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = -\vec{i} - 3\vec{j} + 4\vec{k}$$

$$\alpha: \quad P_0(2, 3, 0) \quad \vec{N} = (-1, -3, 4)$$

$$-(x-2) - 3(y-3) + 4(z-0) = 0$$

$$-x - 3y + 4z + 11 = 0$$

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Napisati jednačinu ravni  $\alpha$

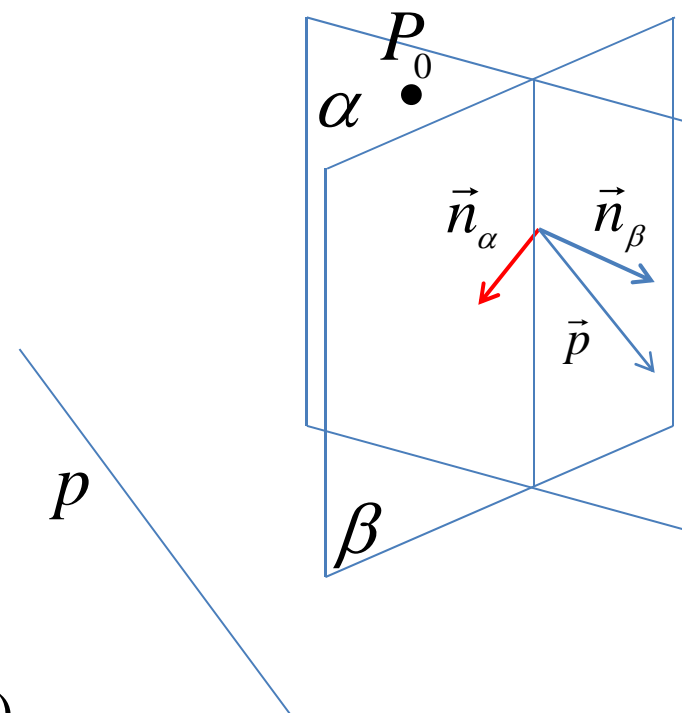
koja sadrži tačku  $P_0(0, 2, 3)$ ,

normalna je na ravan

$$\beta: 2x + y - 2z + 2 = 0$$

i paralelna pravoj

$$p: \frac{x+2}{1} = \frac{y}{-3} = \frac{z}{1} .$$



$$\vec{a} = \vec{n}_\beta = (2, 1, -2) \quad \vec{b} = \vec{p} = (1, -3, 1)$$

$$\vec{N} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & -3 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -5\vec{i} - 4\vec{j} - 7\vec{k}$$

$$\alpha: P_0(0, 2, 3) \quad \vec{N}_\alpha = (5, 4, 7) \quad 5(x-0) + 4(y-2) + 7(z-3) = 0$$

$$5x + 4y + 7z - 29 = 0$$