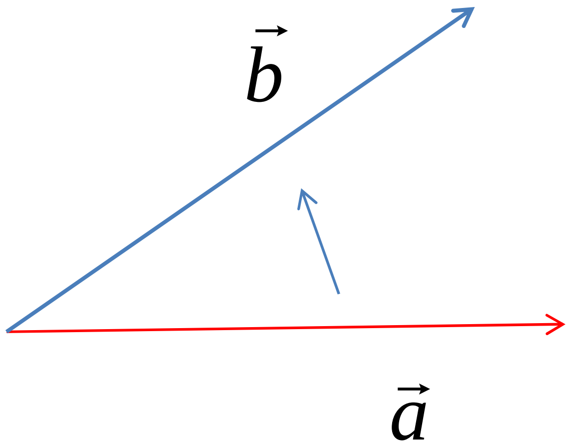
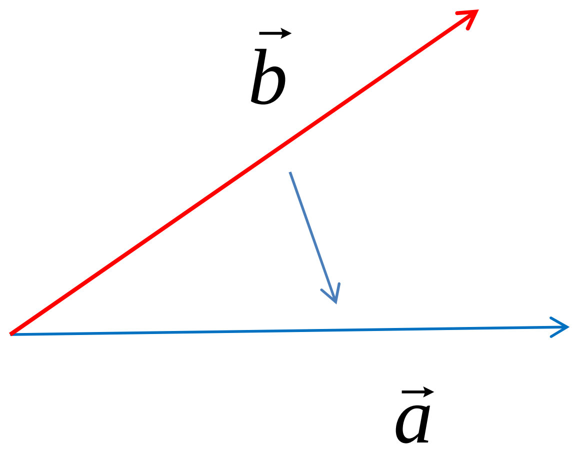
UGAO IZMEDJU DVA VEKTORA

Ugao izmedju dva vektora je ugao za koji treba zarotirati jedan od njih da bi se poklopio po pravcu I smeru sa drugim vektorom.

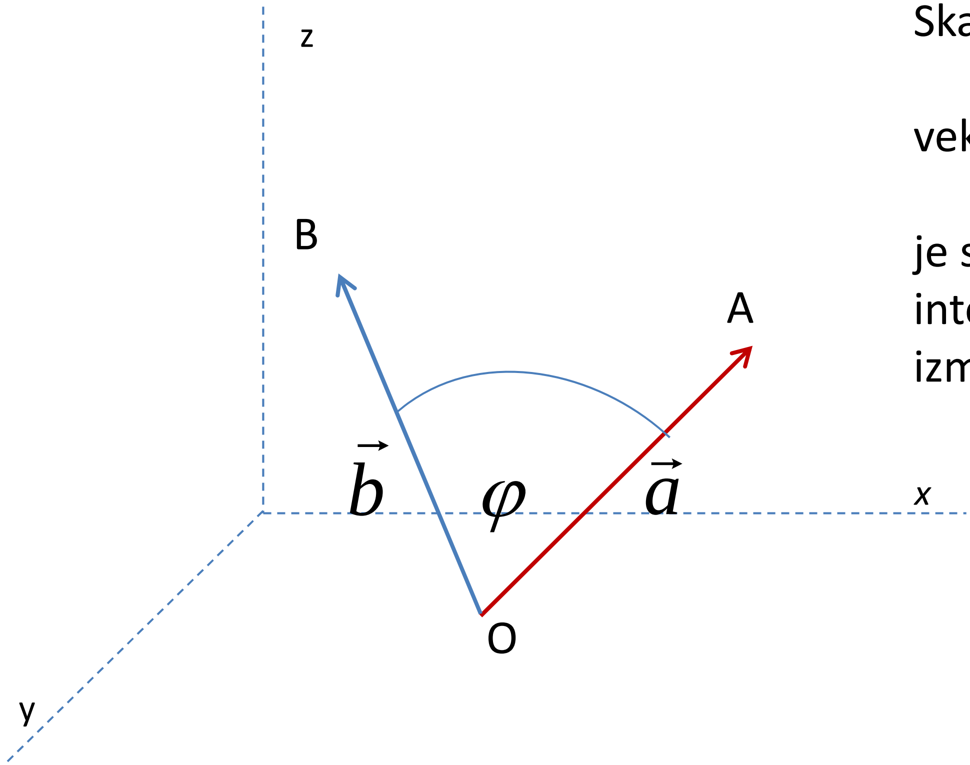
$\angle(\vec{a},\vec{b})$



$\angle(\vec{b},\vec{a})$



SKALARNI PROIZVOD VEKTORA



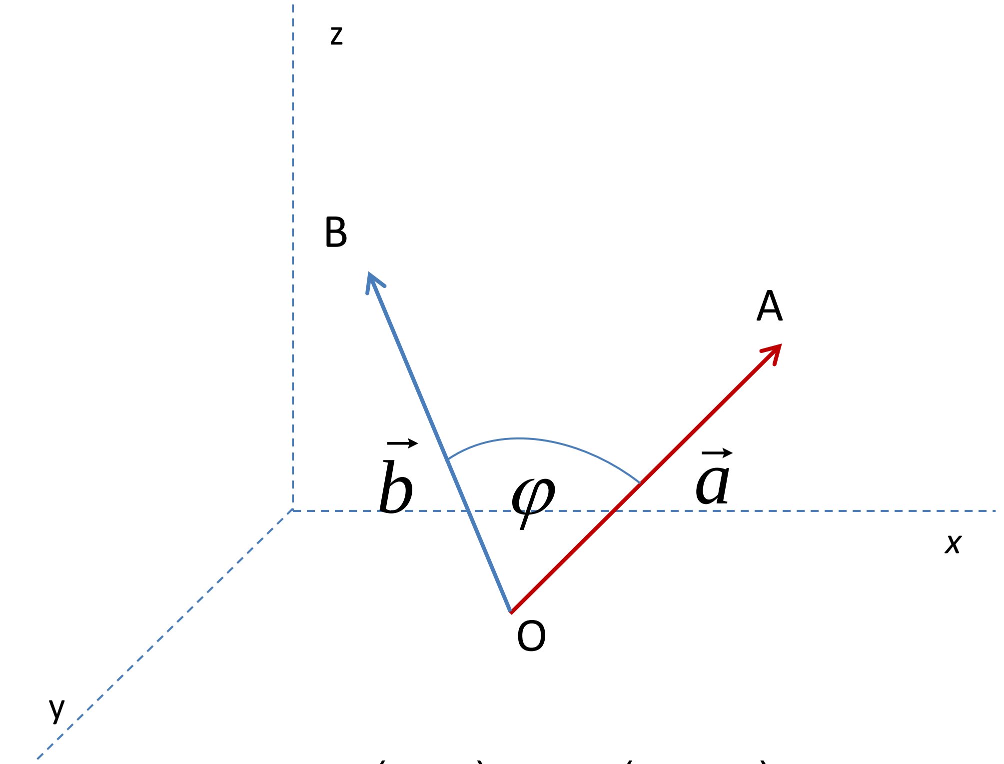
Skalarni proizvod $\vec{a} \cdot \vec{b}$ dva vektora $\vec{a}$ i $\vec{b}$

je skalar, koji je jedanak proizvodu intenziteta vektora i kosinusa ugla izmedju njih.

$\vec{a} \quad \vec{b}$

$$\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \cos\varphi$$

SKALARNI PROIZVOD VEKTORA



$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi$$

Osobine skalarnog proizvoda

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$$

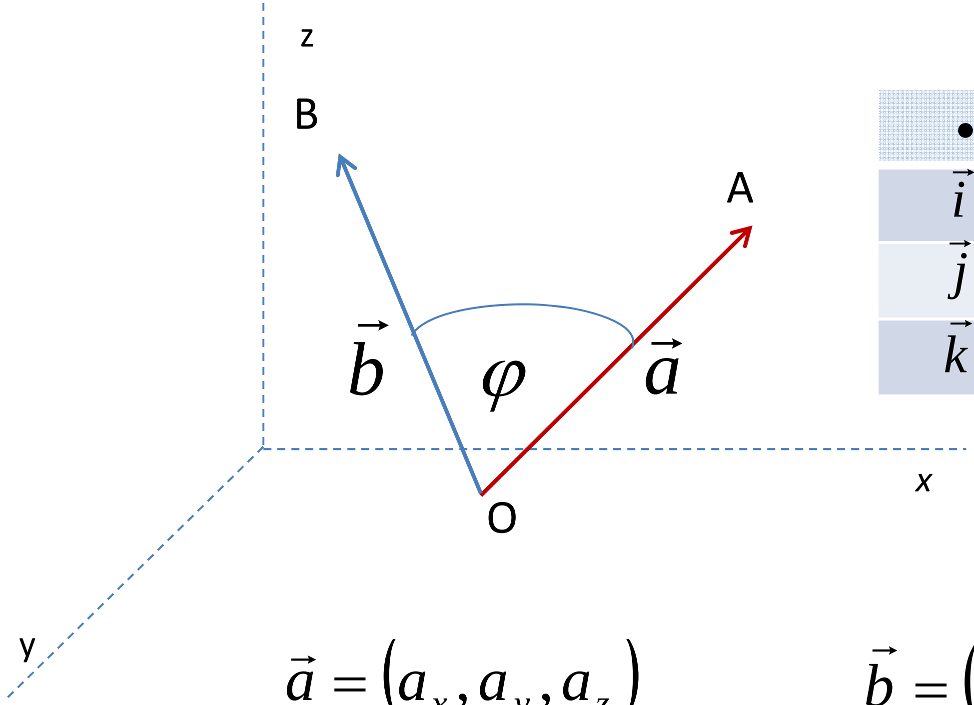
$$(\lambda\vec{a})\cdot\vec{b}=\vec{a}\cdot(\lambda\vec{b})=\lambda(\vec{a}\cdot\vec{b})$$

$$\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

$$\vec{a}\cdot\vec{a}=|\vec{a}|^2=|\vec{a}|^2\geq 0$$

$$\vec{a}\cdot\vec{a}=0 \Leftrightarrow \vec{a}=0$$

SKALARNI PROIZVOD VEKTORA ZADATIH POMOĆU KOORDINATA



$\bullet$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

$$\vec{a} = (a_x, a_y, a_z) \qquad \vec{b} = (b_x, b_y, b_z)$$

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \qquad \vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

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Izračunati skalarni proizvod datih vektora:

$\vec{a} = (1, 1, -2) \qquad \vec{b} = (1, -1, 4)$

$\vec{a} \cdot \vec{b} = 1 \cdot 1 + 1 \cdot (-1) + (-2) \cdot 4 = -8$

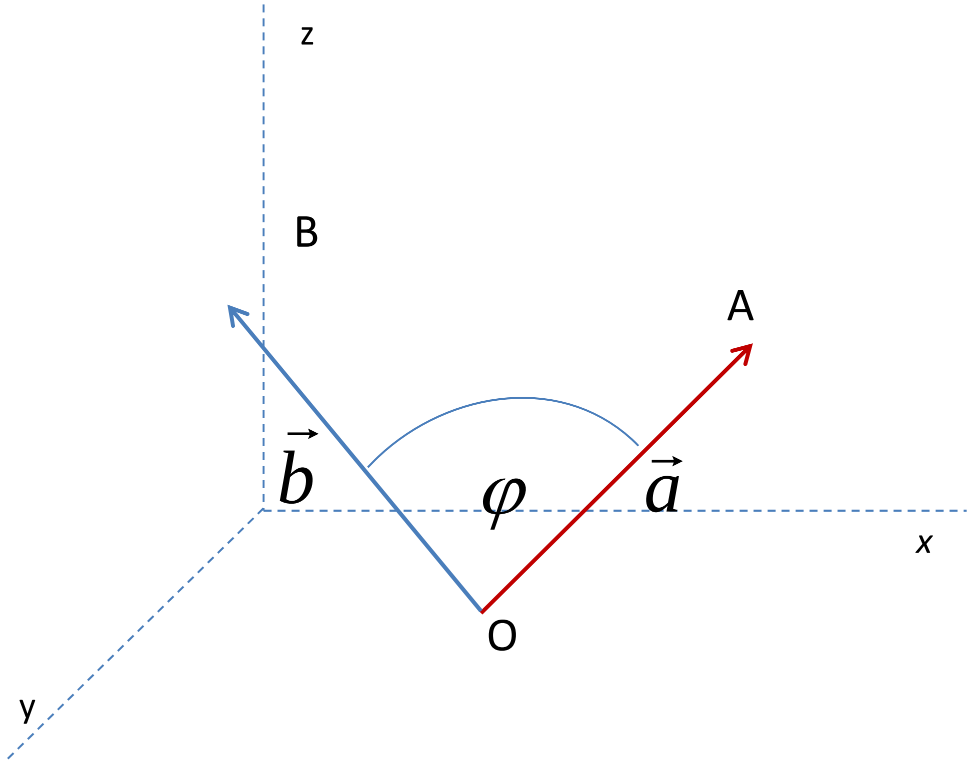
$\vec{a} = (1, 5, -2) \qquad \vec{b} = (1, -1, 4)$

$\vec{a} \cdot \vec{b} = 1 \cdot 1 + 5 \cdot (-1) + (-2) \cdot 4 = -12$

$\vec{a} = (1, -2, -2) \qquad \vec{b} = (1, -1, 3)$

$\vec{a} \cdot \vec{b} = 1 \cdot 1 + (-2) \cdot (-1) + (-2) \cdot 3 = -3$

SKALARNI PROIZVOD VEKTORA - USLOV ORTOGONALNOSTI DVA VEKTORA



$$\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \cos\varphi$$

Dva nenula vektora su ortogonalni ako I samo ako je njihov skalarni proizvod jednak nuli.

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \varphi = 90^0$$

Uslov ortogonalnosti vektora zadatih pomoću koordinata

$$\vec{a} = \left(a_x, a_y, a_z\right) \qquad \vec{b} = \left(b_x, b_y, b_z\right)$$

$$\boxed{\vec{a} \cdot \vec{b} = 0 \Leftrightarrow a_x b_x + a_y b_y + a_z b_z = 0}$$

PRIMER

Vektori

$\vec{a} = (-3, 3, 2) \qquad \vec{b} = (4, 6, -3)$

Su ortogonalni zato sto je

$$\vec{a} \cdot \vec{b} = (-3) \cdot 4 + 3 \cdot 6 + 2 \cdot (-3) = -12 + 18 - 6 = 0$$

Vektori

$\vec{a} = (-3, 3, 2) \qquad \vec{b} = (4, 6, -2)$

nisu ortogonalni zato sto je

$$\vec{a} \cdot \vec{b} = (-3) \cdot 4 + 3 \cdot 6 + 2 \cdot (-2) = -12 + 18 - 4 = 2 \neq 0$$

PRIMER

Odrediti vrednost parametra λ tako da su vektori $\vec{a} = (\lambda, 3, 2)$ i $\vec{b} = (4, 6, \lambda)$ ortogonalni.

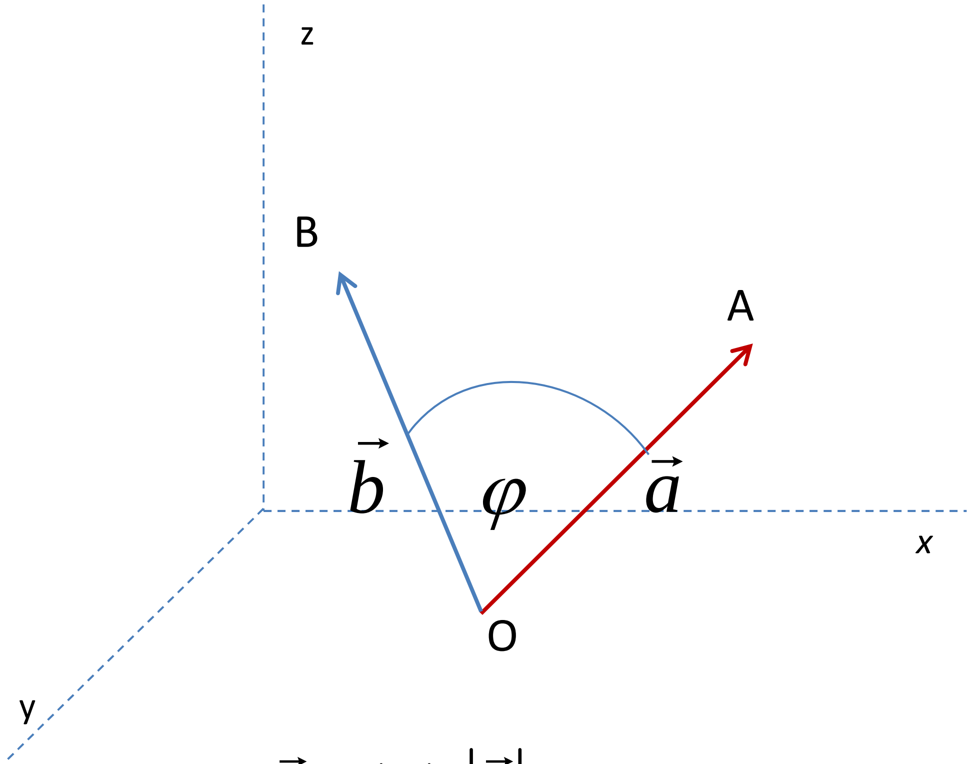
$$\boxed{\vec{a} \perp \vec{b} \;\Leftrightarrow\; \vec{a} \cdot \vec{b} = 0}$$

$$\vec{a} \cdot \vec{b} = \lambda \cdot 4 + 3 \cdot 6 + 2 \cdot \lambda = 0$$

$$6\lambda + 18 = 0$$

$$\lambda = -3$$

SKALARNI PROIZVOD VEKTORA - UGAO IZMEDJU DVA VEKTORA



$$\vec{a} = (a_x, a_y, a_z)$$

$$\vec{b} = (b_x, b_y, b_z)$$

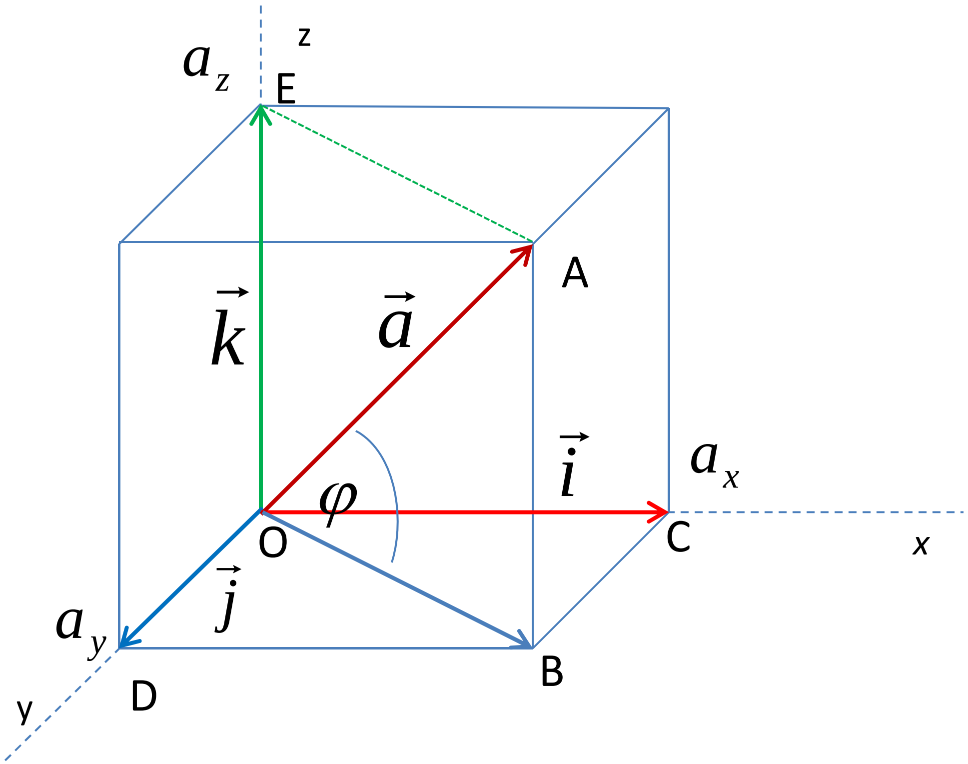
$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi \qquad \cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

$$\cos\varphi = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}\sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2}}$$

PRIMER

Data je kocka osnovne ivice $a = 1$ sa jednim temenom u koordinatnom početku O i sa ivicama koje polaze iz tog temena koje leze na pozitivnim delovima koordinatnih osa. OA je dijagonala te kocke. OB je dijagonala donje osnove te kocke.
Odrediti kosinus ugla izmedju vektora $\overrightarrow{OA}$ i $\overrightarrow{OB}$



$$\vec{a} = \overrightarrow{OA} = (1,1,1)$$

$$\vec{b} = \overrightarrow{OB} = (1,1,0)$$

$$|\vec{a}| = \sqrt{3} \qquad |\vec{b}| = \sqrt{2}$$

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$\cos\varphi = \frac{1\cdot 1 + 1\cdot 1 + 1\cdot 0}{\sqrt{2}\cdot\sqrt{3}} = \frac{\sqrt{6}}{3}$$

PRIMER

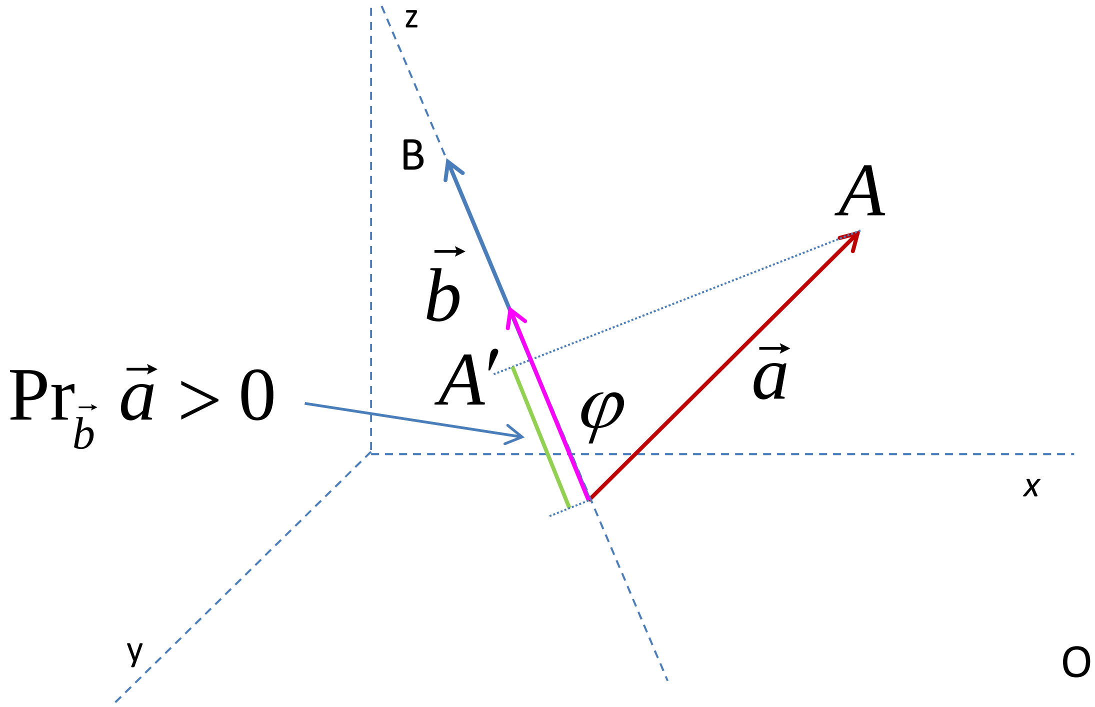
Odrediti ugao izmedju vektora $\vec{a} = (1, 0, 1)$ i $\vec{b} = (0, -1, 1)$.

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

$$\cos\varphi = \frac{1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 1}{\sqrt{1^2 + 0^2 + 1^2} \cdot \sqrt{0^2 + (-1)^2 + 1^2}} = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$$

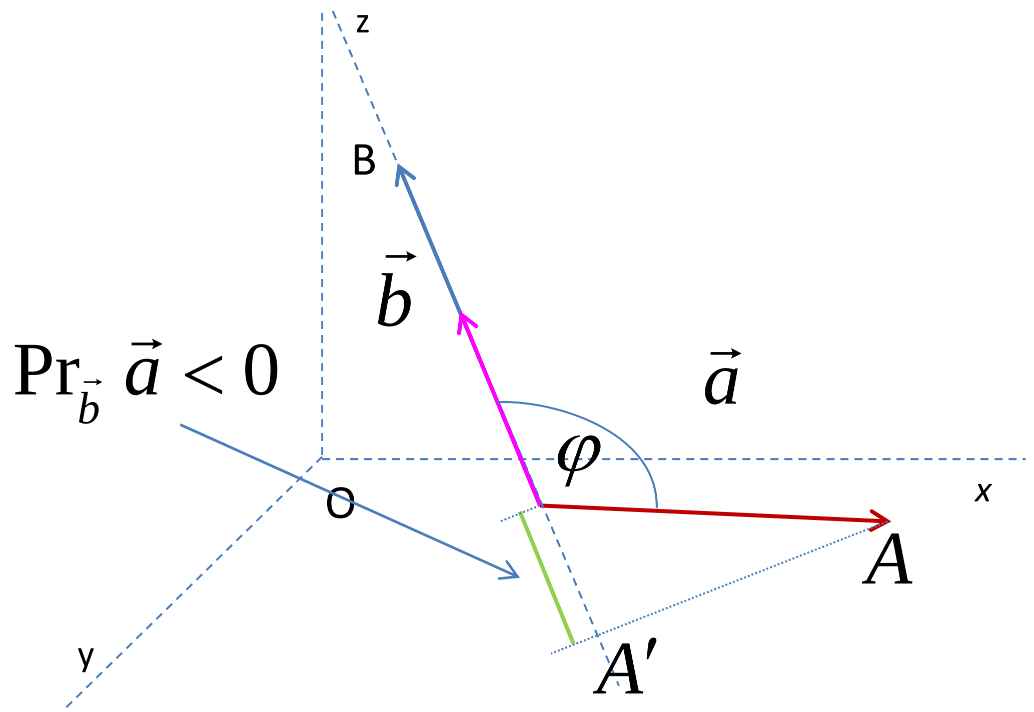
$$\varphi = \frac{\pi}{3} = 60^0$$

ORTOGONALNA ALGEBARSKA PROJEKCIJA VEKTORA



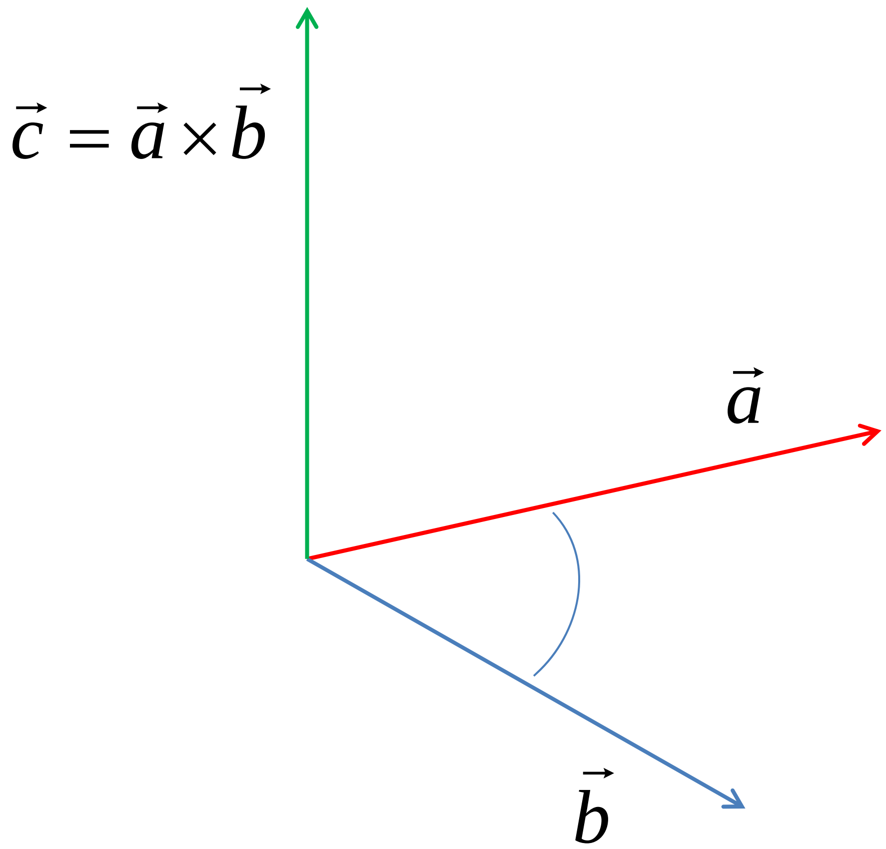
$$ort\,\vec{b} = \frac{\vec{b}}{\left|\vec{b}\right|} \qquad \mathrm{Pr}_{\vec{b}}\,\vec{a} = \left|\vec{a}\right| \cdot \cos\varphi = \vec{a} \cdot ort\,\vec{b}$$

ORTOGONALNA ALGEBARSKA PROJEKCIJA VEKTORA



$$ort\,\vec{b} = \frac{\vec{b}}{\left|\vec{b}\right|} \qquad \mathrm{Pr}_{\vec{b}}\,\vec{a} = \left|\vec{a}\right| \cdot \cos\varphi = \vec{a} \cdot ort\,\vec{b}$$

VEKTORSKI PROIZVOD



Vektorski proizvod vektora $\vec{a}$ i $\vec{b}$ je vektor

$$\vec{c} = \vec{a} \times \vec{b}$$

čiji je intezitet jednak površini paralelograma konstruisanog nad

vektorima $\vec{a}$ i $\vec{b}$,

pravac normalan na ravan odredjenu tim vektorima,

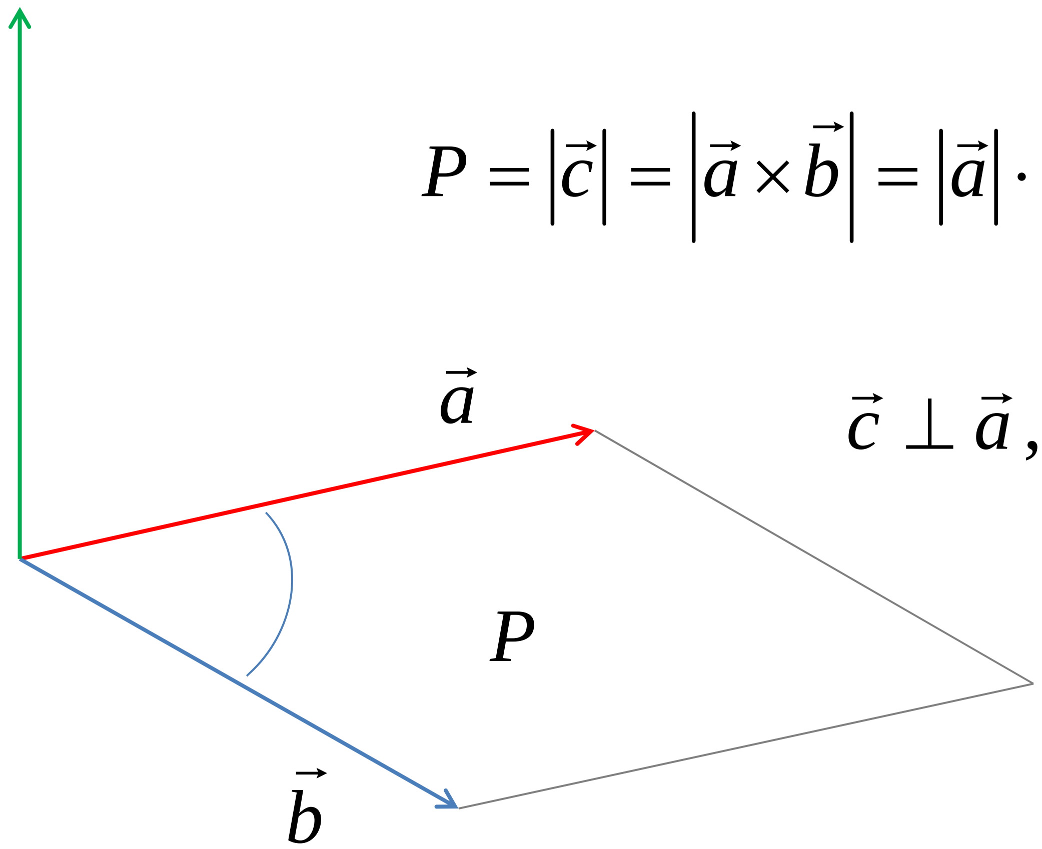
a smer je takav da vektori

$$\vec{a}, \ \vec{b}, \vec{c}$$

čine trijedar iste orjentacije kao i koordinatni sistem.

VEKTORSKI PROIZVOD

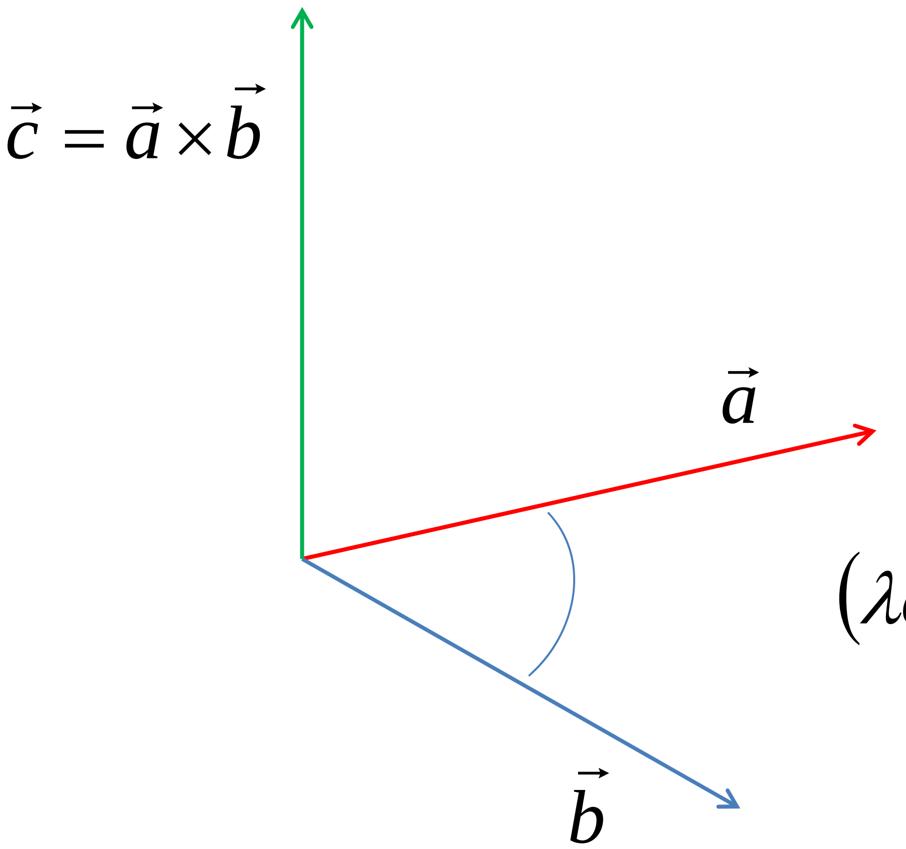
$$\vec{c} = \vec{a} \times \vec{b}$$



$$P = |\vec{c}| = \left|\vec{a} \times \vec{b}\right| = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \sin \angle\left(\vec{a}, \vec{b}\right)$$

$$\vec{c} \perp \vec{a}\,,\ \vec{c} \perp \vec{b}$$

VEKTORSKI PROIZVOD - OSOBINE

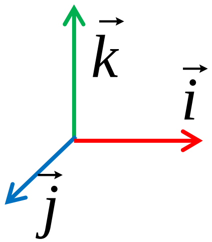


$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

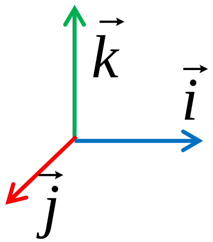
$$(\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) = \lambda (\vec{a} \times \vec{b})$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

VEKTORSKI PROIZVOD - IZRAČUNAVANJE



$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	0	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	0	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	0



$$\vec{a} = \left(a_x, a_y, a_z\right) \qquad \vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$\vec{b} = \left(b_x, b_y, b_z\right) \qquad \vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

$$\vec{a} \times \vec{b} = \left(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\right) \times \left(b_x\vec{i} + b_y\vec{j} + b_z\vec{k}\right)$$

$$\vec{a} \times \vec{b} = \vec{i}\left(a_y b_z - a_z b_y\right) - \vec{j}\left(a_x b_z - a_z b_x\right) + \vec{k}\left(a_x b_y - a_y b_x\right)$$

VEKTORSKI PROIZVOD - IZRAČUNAVANJE

$$\vec{a}\times\vec{b}=\vec{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}-\vec{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}+\vec{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}=\vec{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}-\vec{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}+\vec{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

DETERMINANTE DRUGOG I TREĆEG REDA

Izracunati determinantu, drugog I treceg reda

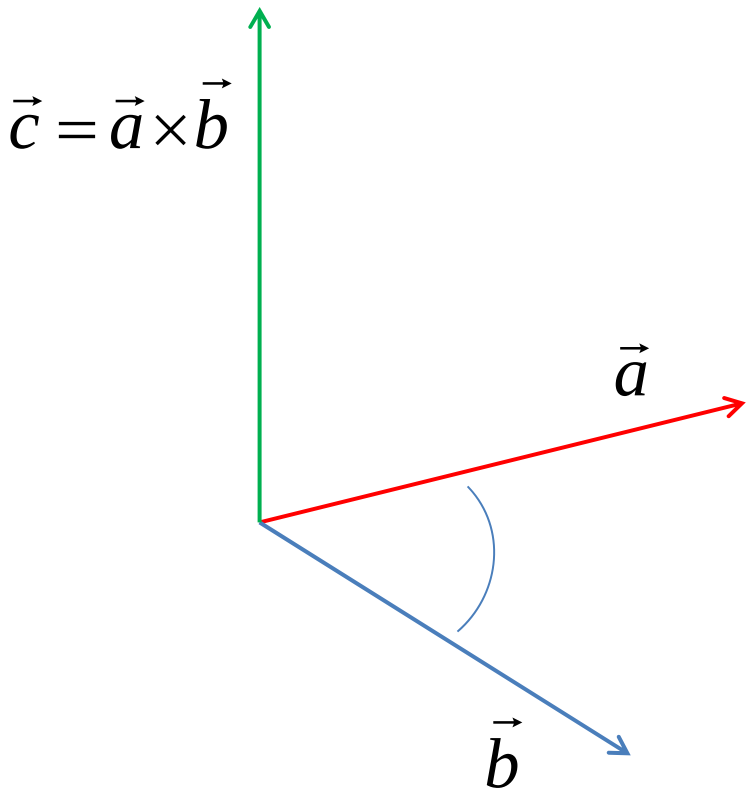
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{22}$$

$$\begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix} = (-1) \cdot 4 - 3 \cdot (-3) = 5$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & -2 \\ -1 & 3 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} -1 & -2 \\ -3 & 1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix} = 22 + 7 - 10 = 19$$

PRIMER



Izračunati vektorski proizvod vektora

$$\vec{a} = (1, -1, 2)$$

$$\vec{b} = (-1, 1, 0)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} == -2\vec{i} - 2\vec{j}$$

$$\vec{a} \times \vec{b} == (-2, -2, 0)$$

PRIMER

Za date vektore $\vec{a} = (1,1,-2)$ $\vec{b} = (1,0,2)$

Izračunati $\vec{a} \times \vec{b}$ i $\vec{b} \times \vec{a}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 1 & 0 & 2 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = 2\vec{i} - 4\vec{j} - \vec{k}$$

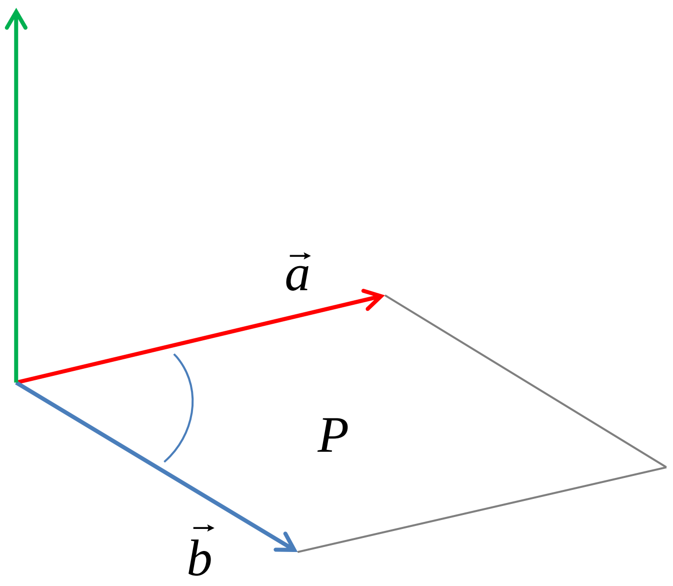
$$\vec{a} \times \vec{b} = (2,-4,-1)$$

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} = -(2,-4,-1) = (-2,4,1)$$

PRIMER

Izračunati površinu paralelograma konstruisanog nad datim vektorima:

$\vec{a} = (1, 5, -2) \qquad \vec{b} = (1, 0, -2)$



$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \vec{i} \begin{vmatrix} 5 & -2 \\ 0 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix}$$

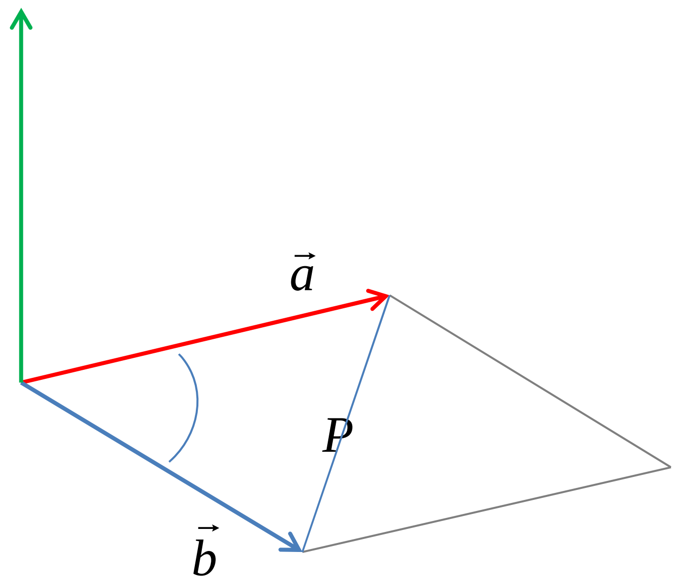
$$\vec{a} \times \vec{b} = 10\vec{i} - 5\vec{k} \qquad \vec{a} \times \vec{b} = (10, 0, -5)$$

$$P = \left|\vec{a} \times \vec{b}\right| = \sqrt{10^2 + 0^2 + (-5)^2} = \sqrt{125} = 5\sqrt{5}$$

PRIMER

Izračunati površinu trougla konstruisanog nad datim vektorima:

$\vec{a} = (1, 5, -2) \qquad \vec{b} = (1, 0, -2)$



$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \vec{i} \begin{vmatrix} 5 & -2 \\ 0 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix}$$

$\vec{a} \times \vec{b} = 10\vec{i} - 5\vec{k} \qquad \vec{a} \times \vec{b} = (10, 0, -5)$

$$P = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{10^2 + 0^2 + (-5)^2} = \frac{1}{2}\sqrt{125} = \frac{5\sqrt{5}}{2}$$