# VISUALIZING PROPERTIES OF A QUADRATIC FUNCTION USING TORRICELLI'S FOUNTAIN

Milena Davidović[1], Tatjana Marković Topalović[2], Mirjana Božić[3], Josip Sliško[4]
1 Faculty of Civil Engineering, University of Belgrade, Belgrade, Serbia,
email: *milena@grf.bg.ac.rs*

2 Dr Andra Jovanović Medical School, Šabac, Serbia
3 Institute of Physics, University of Belgrade, Belgrade, Serbia
4 Facultad de Ciencias Físico Matemáticas, Mexico city, Mexico

In the same chapter of his book *Opera geometrica,* Torricelli published[1] two discoveries: 1) initial velocity of a jet from a container increases with the square root of the depth of the hole¸ 2) he draw the pattern of jets from three openings at the wall of a container filled with water to constant level $H$ and determined the height of the hole with maximal range. In studying the pattern Torricelli used the mentioned law of initial velocities and Galileo's law of free fall and projectile motion. The first Torricelli's discovery is now well known in physics education under the name Torricelli's law. But the pattern of jets from a container entered into physics literature along two ways, which we propose to name: "da Vinci's way" and "Torricelli's way". Along "da Vinci's way" educators and textbook authors (Ref. 2 and textbooks and articles cited by Biser[3] and Atkin[4]) present incorrect drawings of jets in order to incorrectly "demonstrate"the correct Torricelli's law. Along "Torricelli's way" educators point out[3-11] that the shape and range of a jet depend on the initial velocity as well as on the time of flight of a jet. Using algebra and calculus (instead of geometry, proportions and narrative used by Torricelli and Galileo) the shape of trajectories, their envelope, range and meeting of two jets at an arbitrary datum level, are determined by quadratic function and quadratic equation. Their detailed mathematical analysis is presented in this paper.

In describing how the use of water and air through time has developed our scientific understanding, and how to bring fluid mechanics to the general public, E. Guyon and M. Guyon[12] observed: "Water fountains and jets are still being built and are favorite public attractions but, alas, are seldom connected to their scientific meaning, unlike the Torricelli fountain shown in Fig. 1."

## 1. The interplay of mathematics and physics in physics education

It is well known that mathematics and physics are strongly interrelated in a fruitful relationship. For centuries the use of mathematics is an important part of the methodology of physics. For researchers on the role of mathematics in physics education the starting points are:

“Mathematics is the backbone of physics. It provides a language for the concise expression and application of physical laws and relations… As physics teachers, we share a responsibility to help our students develop fluency with the mathematics of physics,” wrote Bing and Redish.[14] “Mathematics is more than just a tool for working with physics problems: the discourse in physics is mathematical in nature.”[15]

We are going to demonstrate how, the application of mathematical knowledge about quadratic function and quadratic equation, offers a unified view on the presentations and discussions[3-11,13,16,17], in the physics textbooks and physics education journals, about jets from Torricelli’s fountain.

**Fig. 1.Torricelli's fountain[13] located in the Center for Advancement of Educators in Šabac, Serbia, exemplifies well the parabolic jet pattern found by Torricelli.**

## 2. Torricelli’s law and jets’ pattern

In his book *Opera Geometrica*, Torricelli dedicated[1] one chapter to the motion of water. In this chapter he studied the discharge (efflux) of water from holes at the wall of a cylindrical vessel. At Fig. 2, taken from this chapter, $\overline{AB}$ represents the wall of a vessel and E, D, C mark the posititions of holes. Torricelli’s first goal was to determine the velocity of efflux of a jet from a hole, let it be hole E. Assuming that there is no air resistence to the movement, he came to the conclusion[1,18,19] that: The speed of a jet at a point is equal to the speed that a single drop of liquid would have if it could fall freely in the vacuum from the liquid level above the orifice”. This is today known as Torricelli’s law[5,11,18] which is written in the form:

$$v_{x0} = \sqrt{2g\overline{AE}} \quad (1)$$

where $g$ is gravitational acceleration.

In contemporary physics courses and journals, equation (1) has been usually derived from Bernoulli's equation assuming that the liquid flow is non-viscous and that air resistance is negligible. This equation is written at two levels, corresponding to water surface and the opening:

$$\frac{\rho v_{x0}^2}{2} = \rho g \overline{AE} \qquad (2)$$

In the next step, Torricelli states that the fluid jet will have a parabolic shape and detemined the range $\overline{GB}$ of this jet at the level of vessel's bottom. In his reasoning Torricelli refers[1] to Galileo's theory of projectile motion[20], without citing any specific Galileo's publication. We should have in mind that Torricelli was assistant of Galileo during last three months of Galileo's life and was appointed his successor as the grand-ducal mathematician and chair of mathematics at the University of Pisa[21].

Adhering to the mathematical standards of doing physics during a large part of 17th century, neither Galileo nor Torricelli used algebra, derivatives and integrals. Theories of Galileo[20] and Torricelli[1]were based on the results of some experiments, on geometric rules and proportions, discussed in Latin.[18,21]Torricelli's drawing,[1]supporting his reasoning about the range of jets from a vessel, is reproduced at Fig. 2, as well as in Refs.19. It is analogous to the drawing in Galileo's unpublished manuscript[22,23] about his experiments on projectile motion (see Figs 7 and 8 in Ref 24) .In fact, Torricelli noted and used the analogy between two motions: a) the motion of a droplet from an orifice at the vessel and b) motion along an inclined plane continued with horizontal section of finite length and free fall from the end of this horizontal section, studied by Galileo[22-24]. In this way, Torricelli was able to state: at the bottom of the vessel the range of a jet from an arbitrary opening $E$ is equal twice the length $\overline{EI}$. It follows that the same range has the jet emerging from the symmetrical opening C. This implies that the range of the central jet is maximal and is equal to the height of the column of water.

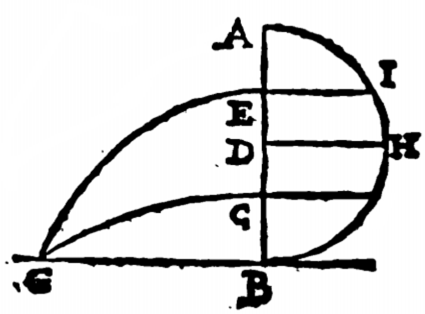


**Fig. 2. The half circle with diameter $\overline{AB}$and jets from a vertical vessel, drawn by Torricelli.[1] *AB* denotes the surface of a vessel with holes. The level of water is kept constant at the level *A*.**

The derivation of the above Torricelli's statements, using geometry, algebra and calculus,would go as follows. We start with Pythagoras theorem for the triangle ΔEDI:

$$\overline{ED}^2 + \overline{EI}^2 = \overline{AD}^2 \tag{3}$$

From (3) and the evident geometric relations at Fig. 2, it follows:

$$\overline{EI} = \sqrt{(\overline{AD} - \overline{ED})(\overline{AD} + \overline{ED})} = \sqrt{\overline{AE}\,\overline{EB}} \tag{4}$$

The time of free fall of a droplet from the opening *E* is given by the equation of a free fall

$$t = \sqrt{2\overline{EB}/g} \tag{5}$$

The motion along the horizontal direction is with the constant velocity. Therefore, the range of a droplet at the bottom level is:

$$\overline{GB} = v_{x0}t = 2\sqrt{\overline{EB}\,\overline{AE}} = 2\overline{EI} \tag{6}$$

in agreement with Torricelli's statement.

## 3. The equation of a droplet trajectory

In order to perceive and generalize Torricelli's derivation, by applying algebra and calculus used in physics education, at Fig. 3 we draw the (x,y) coordinate system and write the equation of a droplet trajectory using the symbols defined at this figure. The emerging droplets are subjected to thegravitational field and form a continuous jet. The components of acceleration are:

$$a_x = 0\,,\ a_y = -g\,, \tag{7}$$

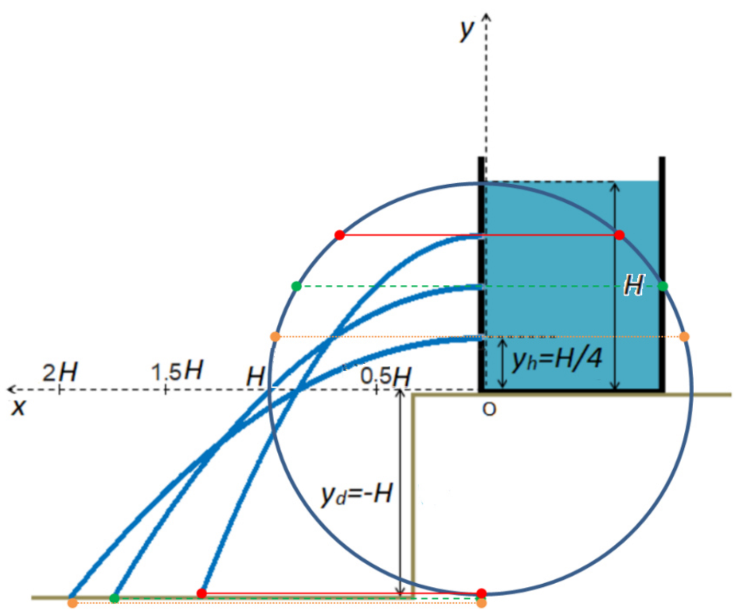

**Fig.3. In the coordinate system (*x*,*y*) trajectories of droplets from three holes at heights $y_h = H/4$, $y_h = H/2$ , $y_h = 3H/4$ are drawn. The datum level is $y_d = -H$. Apart from the trajectories, a circle with diameter along the edge of the vessel, tangent to water surface and ground plane is constructed. It is interesting to note that the range of a jet is equal to the length of the horizontal cord, passing through the emerging hole of the jet. This will be discussed in detail in section 5.**

Assuming that an element of the fluid at $t = 0$ is at the opening, its coordinates at the moment $t$ are[11]:

$$x(t) = v_{x0}t,\ y(t) = y_h - gt^2/2, \tag{8}$$

where the initial velocity using new notation is given by:

$$v_{x0} = \sqrt{2g(H - y_h)} \tag{9}$$

By eliminating time $t$ from Eqs. (8) and (9), one finds the equation of a trajectory of a droplet (the equation of a jet) :

$$y(x) = y_h - x^2/4(H - y_h)\ . \tag{10}$$

By analyzing the quadratic function $y(x)$, for various values of the parameter $y_h$, we are going to describe various physical properties of jets. At Fig. 3 are plotted three functions/trajectories up to the datum level, $y_d$, which is different from the datum level $y_d = 0$. This is because we want to emphasize the perception, prevalent among modern educators[11] and text book writers, that the range of a jet depends on the chosen datum level, i.e. on the position of a tray. As it will be shown, at and below the datum level $y_d = -H$ , the ranges of water jets are in the same order as the depths of the holes. This is the reason for our choice of the datum level at Fig. 3.

We see from (10) that the shape of the trajectory is independent of the gravitational acceleration $g$. This property may be understood starting from the differential equation of the trajectory

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_y}{v_x} \tag{11}$$

where $v_y$ and $v_x$are *y*-and x- components of velocity at point (*x*,*y*). From (8) it follows:

$$v_x = v_{x0}, v_y = -\sqrt{2g(y_h - y)} \tag{12}$$

Therefore:

$$\frac{dy}{dx} = -\frac{\sqrt{y_h - y}}{\sqrt{H - y_h}} \tag{13}$$

By integrating the latter equation one finds Eq. (10). So, we may conclude that the shape of the trajectory does not depend on $g$ because it is determined by the ratio of two velocities of free fall, $v_y$ and $v_x$ . $v_y$ is the velocity of free fall from the hole to the height with coordinate $y$ (the travelled distance $y_h - y$). $v_x$ is the velocity of free fall from the water surface to the height of the hole (the distance $H - y_h$).

## 4. The range of a jet

We are interested how the range of a jet, $x_d$ , at the chosen level $y_d$ , depends on the height of the opening $y_h$. From (10) we find that $x_d^2(y_h)$ is a quadratic function of $y_h$ :

$$x_d^2(y_h) = -4y_h^2 + 4y_h(H + y_d) - 4Hy_d \quad (14)$$

where the interval $y_h \epsilon [0, H]$ is of physical interest. Outside of this interval there is no water. The function $x_d^2(y_h)$ is positive. It is presented at Fig. 4 for five characteristic values of $y_d$, both for positive as well as for negative values of $y_h$. The portions of parabolas for $y_h < 0$ are dotted.

One notes that all parabolas intersect $y_h$ axis at $y_h \equiv y_{h,1}^0 = H$. This is physically understandable. At the top, the initial velocity is zero and therefore the range is zero, for any datum level.

The second point of intersection of a parabola depends on $y_d$, and it is $y_h \equiv y_{h,2}^0 = y_d$. This is also physically understandable. From the opening $y_h = y_d$ time of flight to $y_d$ is zero, consequently the range is zero.

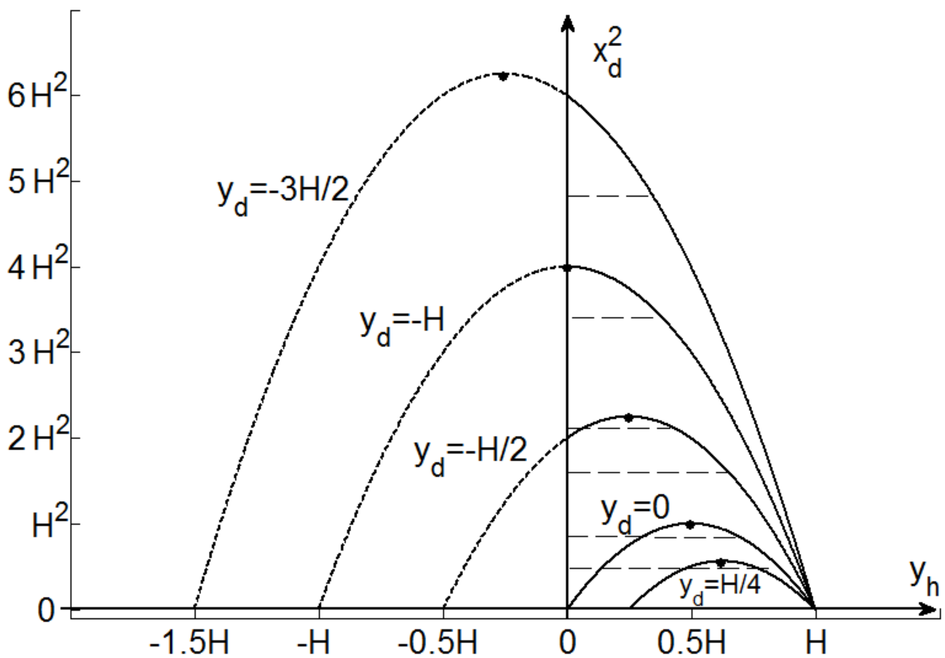


**Fig. 4.The family of parabolas representing the function $x_d^2(y_h)$ .**

The value of datum level $y_d = -H$ is of particular interest, because the maximum of the corresponding parabola (14) lies on the $x_d^2(0)$ axis, as seen at Fig. 4. This maximum has the value

$$x_d^2(0) = 4H^2 \quad , \; x_d(0) = 2H \quad (15)$$

For levels $y_d < -H$ the maximum of the parabola lies in the interval $y_h$<0, where there is no water. Therefore, at the levels $y_d \leq -H$ , the range of a jet from the bottom is always maximal. Maximal range is determined by the intersection of the parabola with the $x_d^2(0)$ axis, i.e.

$$x_{d,max} = x_d(0) = 2\sqrt{-Hy_d}, \, y_d \leq -H \, . \quad (16)$$

More complete analysis of the graphs at Fig. 4 is presented in the Online Appendix.[25]

## 5. Jets' ranges and Torricelli's half circles for arbitrary datum level

Using the quadratic function in Eq. (14), we may generalize Torricelli's method for the determination of jet's range to be applicable for any datum level $y_d$ . For this sake let us rewrite Eq. (14) in the form:

$$\left(\frac{x_d}{2}\right)^2 + \left(y_h - \frac{H+y_d}{2}\right)^2 = \left(\frac{H-y_d}{2}\right)^2 \tag{17}$$

This is the equation of a circle, where $x_d/2$ and $y_h$ are the coordinates of points lying along a circle. The radius and center of the circle are: $r = \frac{H-y_d}{2}$ and $C = \left(0, \frac{H+y_d}{2}\right)$, respectively. At Figs. 3 and 5 are drawn trajectories and half circles for five values of $y_d$ chosen in five characteristic intervals: $y_d > 0$ , $y_d = 0$ , $-H < y_d < 0, y_d = -\mathrm{H}$, $y_d < -H$.

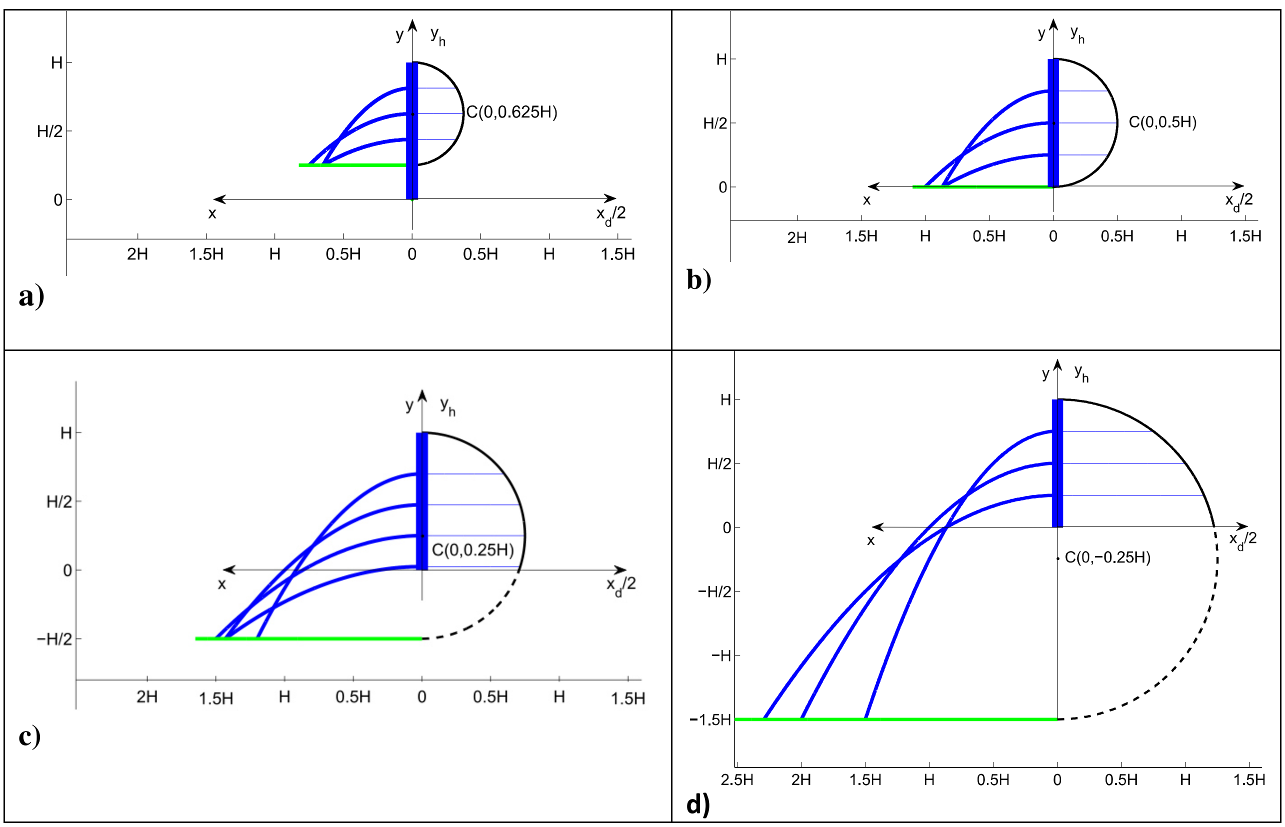


**Fig.5. Generalization of Torricelli's scheme from Fig. 2 to additional datum levels $y_d$: a) $y_d = H/4$ , b) $y_d = 0$ , c) $y_d = -H/2$ , d) $y_d = -3H/2$ . The half-circles are in the**

**( $x_d/2, y_h$) coordinate system. The values: $y_d = H/4$ , $y_d = -H/2$ and $y_d = -3H/2$ belong to three characteristic intervals of $y_d$, as explained in the text. The importance of values $y_d = 0$ and $y_d = -H$ (Fig. 3) is also explained above.**

The set of horizontal lines drawn at Fig 5 from the $y_h$ line to any chosen curve, represents the generalization of the set of three horizontal lines at Torricelli's Fig. 2. For $0 \leq y_d < H$ , for each value of $y_h > y_d$ there exists the height $H - y_h$ whose line has the same length. Jets emerging from these two heights meet at the same point at the datum level. The length of the line from the height $\frac{H-y_d}{2}$ is equal to the radius of the circle and corresponds to the maximal range at this datum level. For $y_d = -\frac{H}{2}$ (as well as for any $y_d \epsilon(-H, 0)$ ), near the height $\frac{H-y_d}{2}$ there exist pairs of heights with the same length of lines (same range). But further from the height $\frac{H-y_d}{2}$ there are no such pairs of heights. For $y_d = -2H$(as well as for any $y_d \leq -H$) there are no pairs of heights with the same lengths of lines. With decreasing height of a hole, the length of the line from $y_h$ to the portion of a circle increases. This implies that if we put a tray at a level $y_d \leq -H$ we would observe that jets are in the same order as the depths of the holes.

The dependence of range on the height of a hole at various datum levels has been recently investigated and graphically presented by Lopac[11] in more general case with different bottle shapes: barrel, bucket corrugated vase.

## 6. Summary and conclusions

The usefulness of Torricelli's fountain in teaching applicability in physics of mathematical properties of the quadratic function and of the roots of the quadratic equation is demonstrated in this paper including the Online Appendix.[25] By paying attention to this aspect of Torricelli's fountain, we expect that teachers would contribute to the decrease of erroneous drawings of water jets in the textbooks and journals. In this way, the inspiring argumentation of Budd and Sangwin[26] about 101 uses of a quadratic equation, would be enriched.

It is easy to make photos of real jets, as were presented by Atkin[4], Sliško and Cruz[7],Sliško[8], Planininšič*et al.*,[9] Božić[13], Lopac[11]. The shapes of trajectories and other features found in these real demonstrations show good agreement with shapes and ranges evaluated from Eqs. (10) and (14) and their mathematical analysis. Therefore, Torricelli's fountain should be used to demonstrate the dependence of the shape and range of jets on two factors: 1) increase of pressure with depth of a fluid and corresponding initial velocity (Torricelli's law) and 2) the time of flight of a droplet from the hole to the datum level. With increasing time of flight, the first factor becomes dominant. So,below $y_d = -H$ the ranges of water jets are in the same order as the depths of the holes. Torricelli's fountain is an ideal apparatus to show and visualize the interplay of physics and mathematics. This interplay may be exploited for integrating physics and mathematics teaching and learning.

## Acknowledgments

We acknowledge support from the Ministry of Education, Science and Technological Development of Serbia under projects OI171028, III45016 (MD) and OI171005, III45016 (MB).

## References


1. E. Torricelli, "De motvaqvarvm" in *Opera Geometrica, Libri Dvo* (Amatoris Maffe&Laurentij de Landis, FlorentiaeTypis, 1644) pp. 191-203, http://archive.org/stream/operageometrica00torrgoog#page/n212/mode/2up
2. E. Grimsehl, (1912). *Lerbuch der Physik. Second enlarged and improved edition* (Verlag von B. G. Teubner, Leipzig and Berlin) Fig. 261, p. 239.
3. R. H. Biser, "The water canexplored again," *Phys. Teach.***4,** 304 (1966).
4. K. Atkin, "The great water-jet scandal," *Phys. Ed.* **23,** 137-138 (1988).
5. L. G. Paldy, "The water can paradox," *Phys. Teach.***1,** 126 (1963).
6. K. Atkin," … and nothing but thetruth!," *Phys. Ed.* **24,** 67-68 (1989).
7. J. Sliško and C. Cruz, "Chorrossoprendentes," *Correo del Maestro***16,** 20-24 (1997).
8. J. Sliško, "Repeated errors in physics textbooks: What do they say about the culture of teaching?" in *Physics Community& Cooperation*: *Selected Contributions from the GIREP-EPEC &PHEC 2009 International Conference*, Vol. 2, edited by D. Raine *et al.* (Lulu, Leicester, 2011)) p. 31.
9. G. Planinšič, C. Ucke and L. Viennot, "Holes in a bottle filled with water: Which water-jet has the largest range?" published by the MUSE group in the Physics Education Division (PED) of the European Physical Society (EPS), (2011), http://www.eps.org/members/group_content_view.asp?group=85190&id=187784
10. A. K. Tamuli, "Liquid flow from orifices," *Phys. Ed.* **23,** 190-191 (1988).
11. V. Lopac, "Water Jets from Bottles, Buckets, Barrels, and Vases with Holes," *Phys. Teach*. **53,** 169-173 (2015).
12. E. Guyon and M. Y. Guyon, "Taking Fluid Mechanics to the General Public,"*Annu. Rev.Fluid Mech.* **46,** 1–22 (2014).
13. M. Božić, " Inspiring learning environment: The school as a three-dimensional textbook," *Europhys. News***44** (2), 22-26 (2013).
14. T. J. Bing and E. F. Redish, "Analyzing problem solving using math in physics: Epistemological framing via warrants," *Phys. Rev. Spec. Top. - Phys. Ed. Res.***5** (2), 020108-1- 020108-15 (2009).
15. O. Uhden, R. Karan, M. Pietrocola and G. Pospiech, "Modelling Mathematical Reasoning in Physics Education," *Science & Education,* **21** (4), 485-506 (2012).
16. J. H. Avison, "The whole truth about water jets," *Phys. Ed.* **23,** 265 (1988).
17. J. H. Avison, "The last drop from the water jets," *Phys. Ed.* **24**, 259 (1989).
18. S. R.Bistafa,"First Theoretical Constructions to the Fluid Mechanics Problem of the Discharge," *Advances in Historical Studies***4,** 172-199 (2015).
19. Society for the Diffusion of Useful Knowledge, *Library of useful knowledge. Natural philosophy.I.,* Chapter *Hydraulics* (Baldwin and Cradock,London, 1829) p. 4.
20. G. Galileo, (1638), *Discorsi e DimostrazioniMatematiche, intorno a due nuovescienze* , translated by Henry Crew&Alfonso de Salvio*, Dialogues concerning TWO NEW SCIENCES* (Macmillan, New York, 1914)
21. A. C. Sparavigna, "Teaching Physics During the 17th Century: Some Examples From the Works of Evangelista Torricelli," (2013) https://arxiv.org/ftp/arxiv/papers/1307/1307.4942.pdf
22. G. Bonera, *Galileo oggi,* (1995) (La Goliardica Pavese, Pavia)
23. S. Drake, (1978), *Galileo at Work* (University of Chicago Press, Chicago)
24. L. Borghi, A De Ambrosis, N Lamberti and P Mascheretti, "A teaching-learning sequence on

free fall motion," *Phys. Ed.* 40 (3), 266-273 (2005).
25. Visualizing properties of a quadratic function using Torricelli's fountain, Online Appendix
26. C. Budd and C. Sangwin, "101 uses of a quadraticequation," (2004) https://plus.maths.org/content/os/issue29/features/quadratic/index

# Visualizing properties of a quadratic function using Torricelli's fountain

## Online Appendix

The pattern of water jets from holes in the cylindrical vessel has been the subject of scientific inquiry and educational dispute since the famous (incorrect) drawing[8,9] made by Leonardo da Vinci in XV century. The correct pattern, from the central hole and two holes symmetrically positioned with respect to the center, was firstly found theoretically by Torricelli[1] in 1644. Afterwards it was presented in textbooks and encyclopedias. The apparatus was recently named[13] "Torricelli's fountain". By the end of XIX century, Torricelli's theoretical pattern, was almost perfectly replicated experimentally by mercury jets. However, by the beginning of the XX century, an erroneous pattern of three jets[2], aimed to illustrate the law of the increase of pressure with depth in a liquid, started its life in physics and science textbooks (see for example the textbooks mentioned by Biser[3] and Atkin[4]). Consequently, in educational journals started to appear articles[3-11] denouncing the error and indicating which pattern is closer to the real behavior of jets. Despite this, publications with erroneous drawing and argumentation continue to appear,[16,17,OA1-OA3] as Lopac[11] pointed our recently in this journal.

In this paper, including the Online Appendix, we demonstrate in detail the usefulness of Torricelli's fountain in teaching applicability in physics of mathematical properties of the quadratic function and of the roots of the quadratic equation. More generally, we show that Torricelli's fountain is an ideal apparatus to visualize properties of the quadratic function and quadratic equation, as well as the interplay of physics and mathematics. This interplay may be exploited in integrating physics and mathematics teaching and learning. The necessity for this integration has been widely discussed among physics and mathematics educators.

### A1. The jet with maximal range at a given level

Using quadratic function (14) and graphs at Fig. 4, let us determine the height of the opening for which the jet range at the chosen datum level (position of a tray) is maximal. The maximum of quadratic function (14) is at:

$$y_h = \frac{H + y_d}{2} \tag{OA.1}$$

The corresponding maximal value is:

$$x^2_{d,max}\left(\frac{H+y_d}{2}\right) = (H - y_d)^2 \Rightarrow x_{d,max}\left(\frac{H+y_d}{2}\right) = H - y_d \tag{OA.2}$$

Taking into account that heights of the holes are in the interval , this maximum determines the maximal range only for $H \geq \frac{H+y_d}{2} \geq 0$, i.e. $H \geq y_d \geq -H$ .
It is instructive to check that the points of maxima and the corresponding maximal values at the parabolas presented at Fig. 4 agree with the general formula (OA.1) and (OA.2). The maxima at the levels $y_d = 0$ and $y_d = -H$ are of particular interest. For $y_d = 0$, maximal range $x_{d,max}(H/2) = H$ has the jet from $y_h = H/2$. For $y_d = -H$ maximal range $x_{d,max}(0) = 2H$ has the jet from the opening at $y_h = 0$ .

For levels $y_d < -H$ the maximum of the function $x_d^2(y_h)$ lies in the region $y_h < -0$ and, in the region $y_h > 0$ it is a decreasing function. Therefore, maximal jet's range is determined by the intersection of the parabola with the $y_h = 0$ axis, $x_{d,max} = x_d(0) = 2\sqrt{-Hy_d}$ . Consequently, at the levels $y_d \leq -H$ , the range of a jet from the bottom is always maximal.

## A2. Meeting of two jets

From graphs at Fig. 4 we see that there are three characteristic intervals of values of $y_d$ : $y_d \in [0, H)$ , $y_d \in [-H, 0)$ , $y_d \in (-\propto, -H)$ . The intervals differ by the number of intersections of a horizontal dashed line, drawn below the maximum, with a parabola. In the interval$y_d \in [0, H)$ there are always two intersections. In the interval $[-H, 0)$ there may be one or two intersections. In the interval $(-\propto, -H)$ there exists only one intersection. These intersections are determined from Eq. (14) by writing it as a quadratic equation for $y_h$, for given $x_d$ and $y_d$ :

$$y_h^2 - y_h(H + y_d) + Hy_d + \frac{x_d^2}{4} = 0 \qquad \text{(OA.3)}$$

By determining the roots of Eq. (OA.3) we can determine the heights from which two jets reach the same point at the given level.

$$y_{h1,h2} = \frac{H + y_d \pm \sqrt{(H - y_d)^2 - x_d^2}}{2} \qquad \text{(OA.4)}$$

As usual, these roots satisfy Vieta's formula:

$$y_{h1} + y_{h2} = H + y_d \qquad \text{(OA.5)}$$

Of physical interest are roots which satisfy $0 \leq y_{hi} < H$ . One may analyze algebraically the roots (OA.4) to find in which intervals of $y_d$ and $x_d$ lie positive roots. Another way is to contemplate graphs at Fig. 4. Let us start with graphs for levels $0 \leq y_d < H$. We see that for each point in the interval $x_d \in (0, x_{d,max})$ there exists two heights $y_h$ from which the jets meet at the level $y_d$. In the case of levels $-H \leq y_d < 0$ there are two characteristic intervals of $x_d$. To each point in the interval $x_d \in \left(2\sqrt{-Hy_d}, H - y_d\right)$ arrive two jets from two heights. To points in the interval $x_d \in (0, 2\sqrt{-Hy_d})$ only one jet arrives. At a level satisfying $y_d < -H$, single jet arrives to each point in the interval $x_d \in (0, H - y_d)$.

At Fig. OA1 is presented the set of trajectories reaching the level $y_d = -H/2$. One clearly sees the interval of points to which arrive two jets. The jet having maximal range is the limiting jet of these pairs of jets.

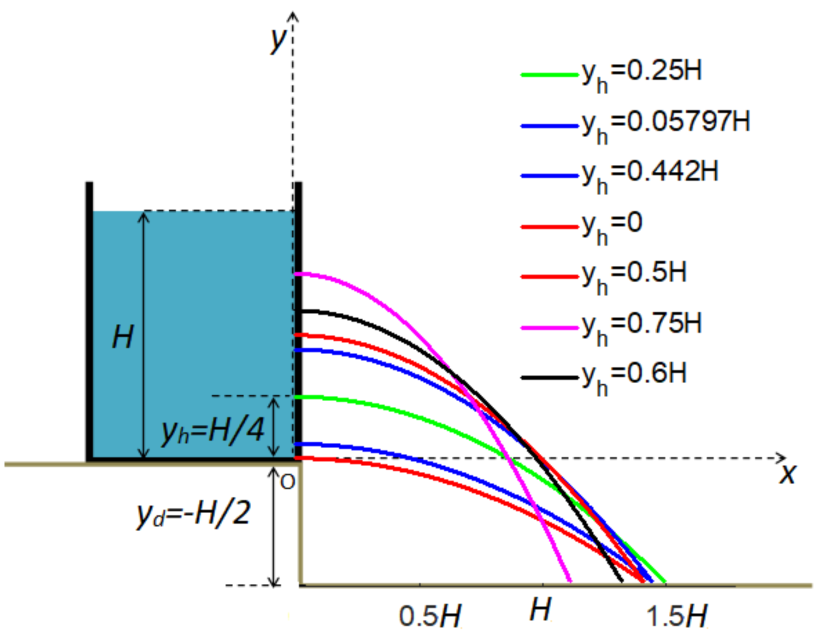


**Fig. OA1. Trajectories from various heights reaching level $y_d = -H/2$. The jet from $y_h = H/4$ has maximal range and it is theright limiting point of the interval of points to which two jets arrive. The left limiting point of this interval, $x_{d,l}$, is determined by the intersection of the parabola at Fig. 4 with $y_h = 0$ axis, so that $x_{d.l} = H\sqrt{2}$.**

At Fig. OA2 is presented the set of trajectories reaching the level $y_d = -3H/2$. To all points at this level arrive only one jet. The jet having maximal range starts at the bottom of the vessel ($y_h = 0$).

We may now rewrite Eq. (OA.5) to determine the level at which two jets from given two heights intersect:

$$y_d = y_{h1} + y_{h2} - H \quad \text{(OA.6)}$$

If we consider the depths of openings, $y'_{hi} = H - y_{hi}$, instead of heights of openings, $y_{hi}$, the latter relation reads:

$$y'_d = y'_{h1} + y'_{h2} \quad \text{(OA.7)}$$

where $y'_d = H - y_d$ . In other words, the depth at which two jets intersect is equal to the sum of depths at which are the openings.

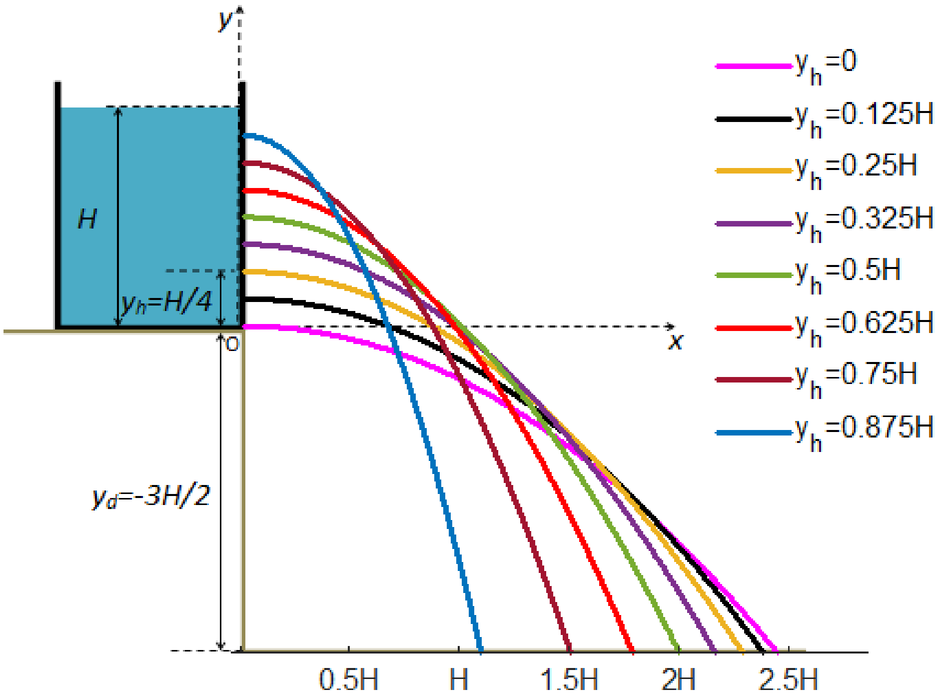


**Fig. OA2.Trajectories reaching the level $y_d = -3H/2$. At this level there are no points to which two jets arrive. The jet from the bottom reaches the greatest distance.**

## A3. Envelope of the family of trajectories

Envelope of a family of curves/trajectories provide additional insight and information about their properties, as shown by Baće *et al.*[OA4], Heppler and Eleuterio[OA5] and other authors. The first example of an envelope curve is due to Torricelli, who showed that a family of ballistic trajectories characterized by having the same initial speed are all tangent at some point to one and the same parabola.[OA5] Torricelli named it "parabola di sicurezza" but it got also the name Torricelli's parabola.[OA5]

Trajectories (parabolas) from Torricelli's tube are defined by Eq. (10). They start at different heights and are characterized by different initial velocities. In order to determine the envelope of this family let us rewrite Eq. (10) in the form

$$4y(H - y_h) = 4y_h(H - y_h) - x^2 , \qquad \text{(OA.8)}$$

and let us introduce the function

$$f(x, y, y_h) \equiv 4yH - 4yy_h - 4y_hH + 4y_h^2 + x^2 = 0 \qquad \text{(OA.9)}$$

The envelope is a curve which is tangent to all curves of the given family. Consequently, it is determined by the following equation:

$$\frac{\partial f}{\partial y_h} = -4y - 4H + 8y_h = 0 \tag{OA.10}$$

and by Eq. (OA.8). From (OA.10) it follows:

$$y_h = \frac{y+H}{2} \tag{OA.11}$$

By substituting (OA.11) into (OA.9) we find the relation:

$$2yH - y^2 - H^2 + x^2 = 0 \tag{OA.12}$$

which leads to:

$$(H-y)^2 = x^2 \tag{OA.13}$$

From the latter relation the equation of the envelope follows:

$$y = H - x \tag{OA.14}$$

We see that the envelope of trajectories from Torricelli's tank is a straight line. At Fig. OA3 is presented the family of trajectories together with the envelope.

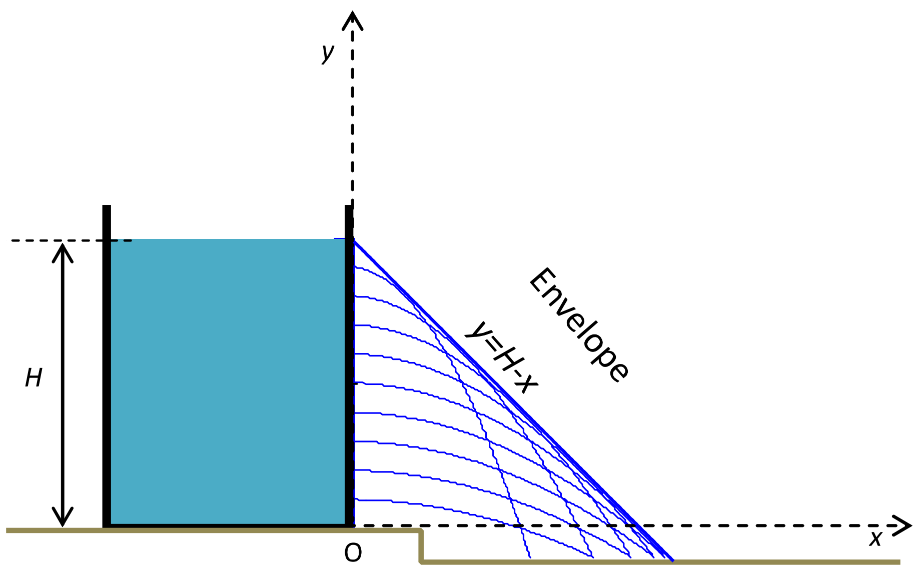


**Fig.OA3. The family of trajectories of jets and their envelope.**

The equation of the envelope of trajectories may be found also by the method, which does not use the calculus. This method, proposed by Baće*et al.*[OA4] , is based on the following properties of the family of trajectories and of the envelope: a) no trajectory on any target plane reaches a range greater than the "range" of the envelope; b) the intersection of the envelope and the target plane gives the maximal range which can be reached by only one trajectory from the family.

In order to apply the Baćeet *al.* method, let us analyze our Eq. (OA.4) which gives two solutions (heights) from which jets arrive to the chosen point ($x_d$,$y_d$). These two solutions are equal if:

$$(H - y_d)^2 - x_d^2 = 0 \quad \text{(OA.15)}$$

i.e. if the following equation is satisfied:

$$H - y_d = x_d \quad \text{(OA.16)}$$

Evidently, the latter equation is the same as Eq. (OA.14) of the envelope, derived above.

By eliminating $x$ from (OA.14) and (OA.9) we may find depth $y_t$ where the envelope touches the jet from $y_h$ .

$$(H - y_t)^2 = 4(H - y_h)(y_h - y_t) \quad \text{(OA.17)}$$

It is easy to check that the solution of (OA.17) is the following one:

$$y_t = 2y_h - H \quad \text{(OA.18)}$$

It is useful to rewrite Eq. (OA.18) using depths (primed quantities) instead of levels. Then, it reads:

$$y'_t = 2y'_h \quad \text{(OA.19)}$$

i.e. the depth of touch is equal to twice the depth of the opening.

**References**


OA1. C. Santamaria, *La physique tout simplement* (Ellipses, Paris, 2007) pp. 17.

OA2. J. H. Avison, *The World of Physics* (Nelson Thornes, Chetenham, UK, 2014) Fig. 6.2

OA3. D. A. Serrano *et al.*, "Trajectories of Water and Send Jets," in *Recent Advances in Fluid Dynamics with Environmental Applications*, edited by J. Klapp *et al.* (Springer International Publishing, Switzerland, 2016) p. 289.

OA4. M. Baće, S. Ilijić, Z. Narančić and L. Bistričić, "The envelope of projectile trajectories, *Eur. J. Phys*. **23,** 637–642 (2002).

OA5. G. R. Heppler and G. *D'Eleuterio,* 'Generalization of Torricelli's parabola to Problems with a moving source," *Am. J. Phys*. **81,** 754 -761 (2013).