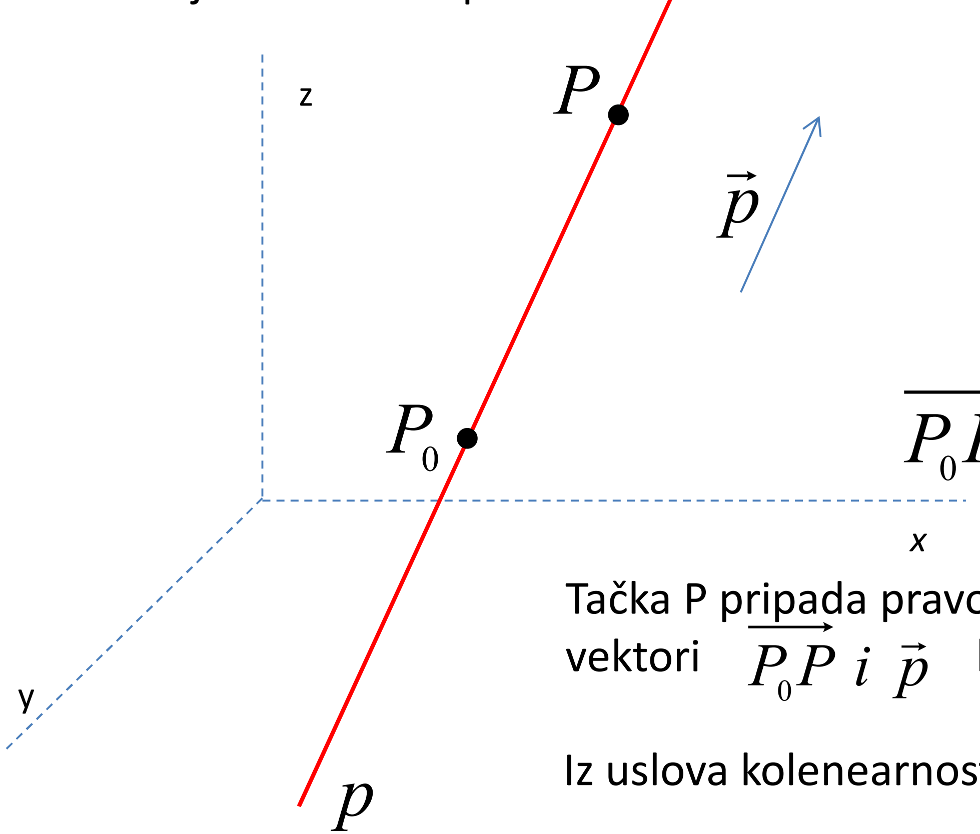
JEDNAČINE PRAVE - KANONSKI OBLIK JEDNAČINA PRAVE

Prava odredjena tačkom i paralelnim vektorom



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

Tačka P pripada pravoj p ako i samo ako su vektori $\overrightarrow{P_0P}$ i $\vec{p}$ kolimearni.

Iz uslova kolenearnosti slede jednačine prave:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

PRIMER

Napisati jednačine prave koja sadrži tačku $P_0(1,-2,3)$

i paralelna je vektoru $\vec{p}=(5,2,-3)$.

$p:$ $P_0(1,-2,3)$

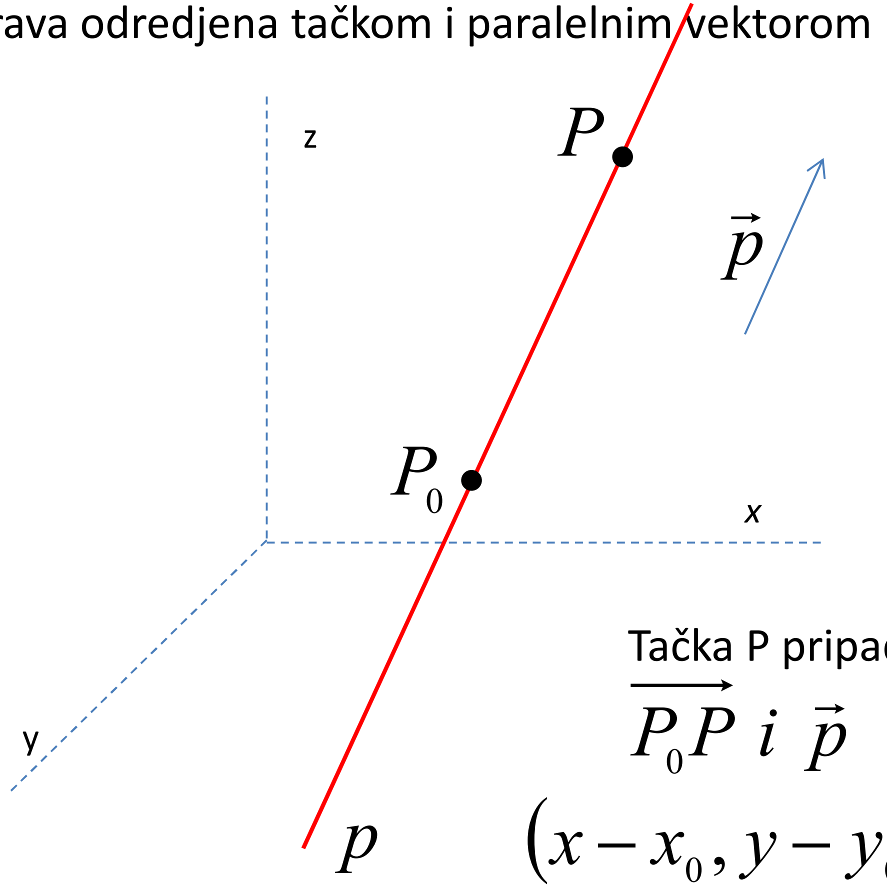
$\vec{p}=(5,-3,2)$

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

$$\frac{x-1}{5}=\frac{y+2}{-3}=\frac{z-3}{2}$$

PARAMETARSKE JEDNAČINE PRAVE

Prava odredjena tačkom i paralelnim vektorom



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

Tačka P pripada pravoj ako i samo ako su vektori

$\overrightarrow{P_0P}$ i $\vec{p}$ kolinearni: $\overrightarrow{P_0P} = t \cdot \vec{p}, \quad t \in R$

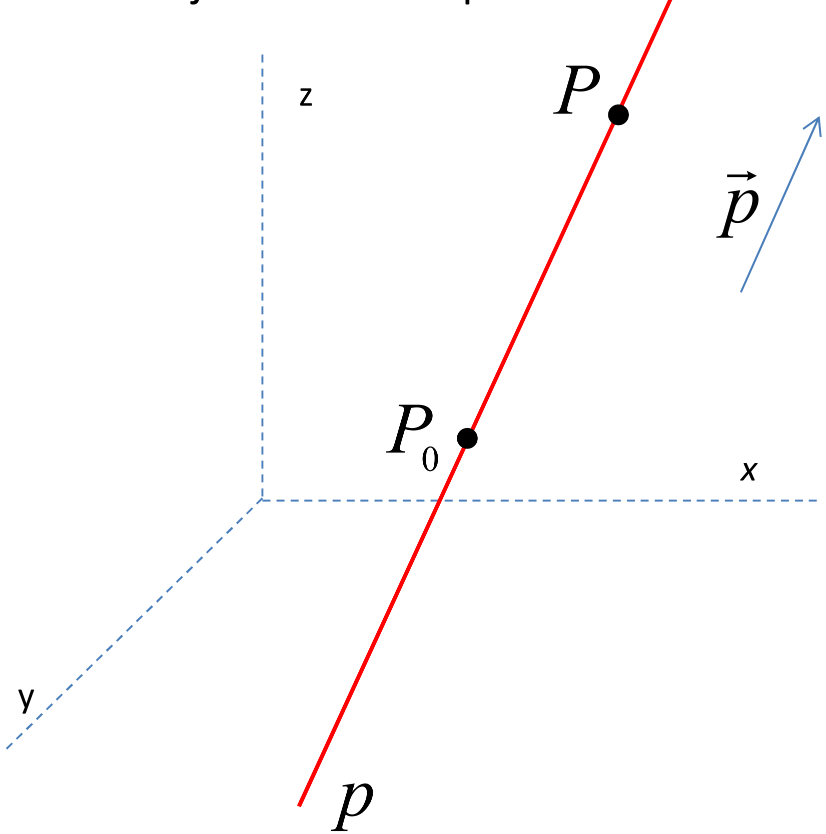
$$(x - x_0, y - y_0, z - z_0) = t(a, b, c), \quad t \in R$$

$$x - x_0 = at, \; y - y_0 = bt, \; z - z_0 = ct, \quad t \in R$$

$$x = x_0 + at, \; y = y_0 + bt, \; z = z_0 + ct, \quad t \in R$$

PARAMETARSKE JEDNAČINE PRAVE

Prava odredjena tačkom i paralelnim vektorom



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

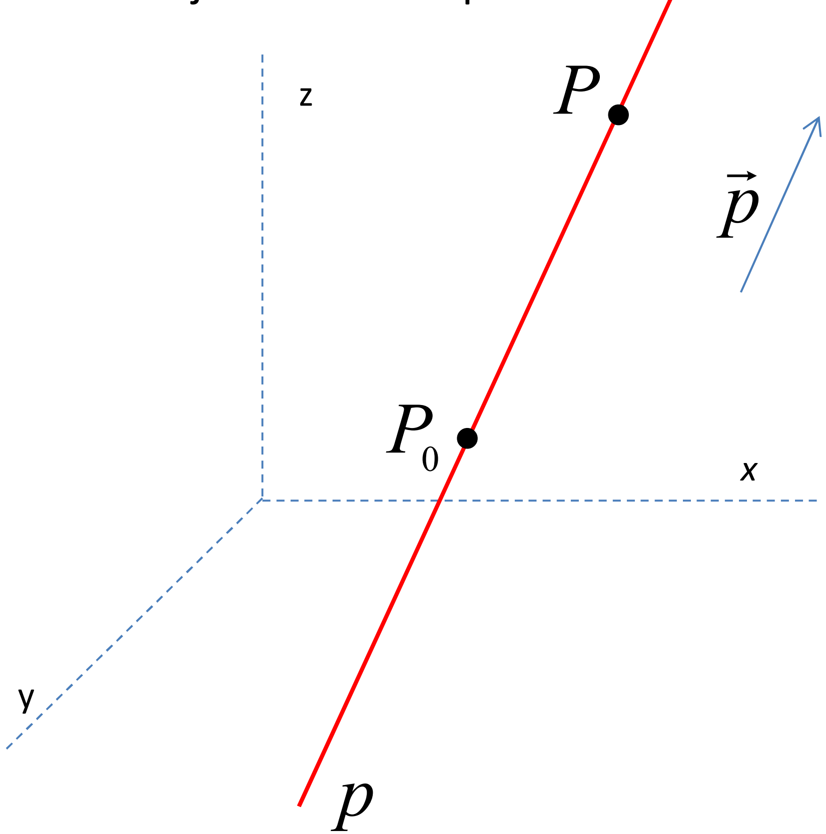
$$x = x_0 + at$$

$$y = y_0 + bt \qquad t \in R$$

$$z = z_0 + ct$$

PARAMETARSKE JEDNAČINE PRAVE

Prava odredjena tačkom i paralelnim vektorom



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = (a, b, c)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

$$x = x_0 + at$$

$$y = y_0 + bt \qquad t \in R$$

$$z = z_0 + ct$$

PRIMER

Napisati parametarske jednačine prave koja sadrži tačku $P_0(1,-2,3)$

i paralelna je vektoru $\vec{p}=(5,-3,2)$.

$$p: \quad P_0(x_0, y_0, z_0)$$

$$\vec{p}=(a,b,c)$$

$$x = x_0 + at$$

$$y = y_0 + bt \qquad t \in R$$

$$z = z_0 + ct$$

$$p: \quad P_0(1,-2,3)$$

$$\vec{p}=(5,-3,2)$$

$$x = 1 + 5t$$

$$y = -2 - 3t \qquad t \in R$$

$$z = 3 + 2t$$

PRIMER*

Napisati parametarske jednačine prave koja sadrži tačku $P_0(1,-2,3)$

i paralelna je vektoru $\vec{p}=(5,-3,2)$.

$$p: \quad P_0(x_0, y_0, z_0)$$

$$\vec{p}=(a,b,c)$$

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}=t$$

$$x=x_0+at$$

$$y=y_0+bt \qquad t\in R$$

$$z=z_0+ct$$

$$p: \quad P_0(1,-2,3)$$

$$\vec{p}=(5,-3,2)$$

$$\frac{x-1}{5}=\frac{y+2}{-3}=\frac{z-3}{2}=t$$

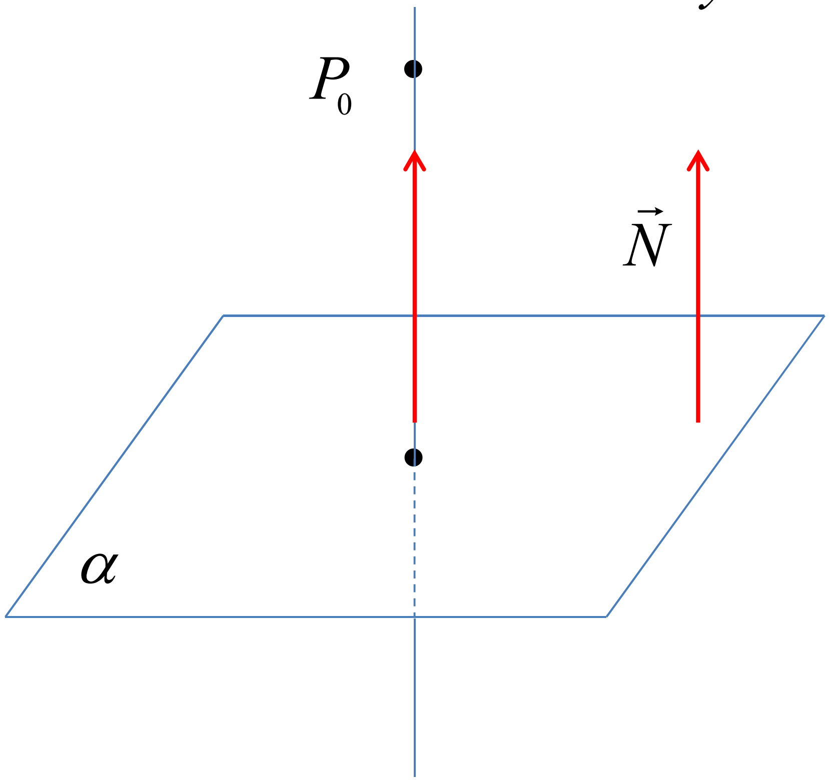
$$x=1+5t$$

$$y=-2-3t \qquad t\in R$$

$$z=3+2t$$

PRIMER

Napisati jednačine prave koja sadrži tačku $P_0(-1,2,-3)$
i normalna je na ravan $-2x-3y+2z+4=0$.



$p:$

$P_0(-1,2,-3)$

$\vec{p}=\vec{N}=(-2,-3,2)$

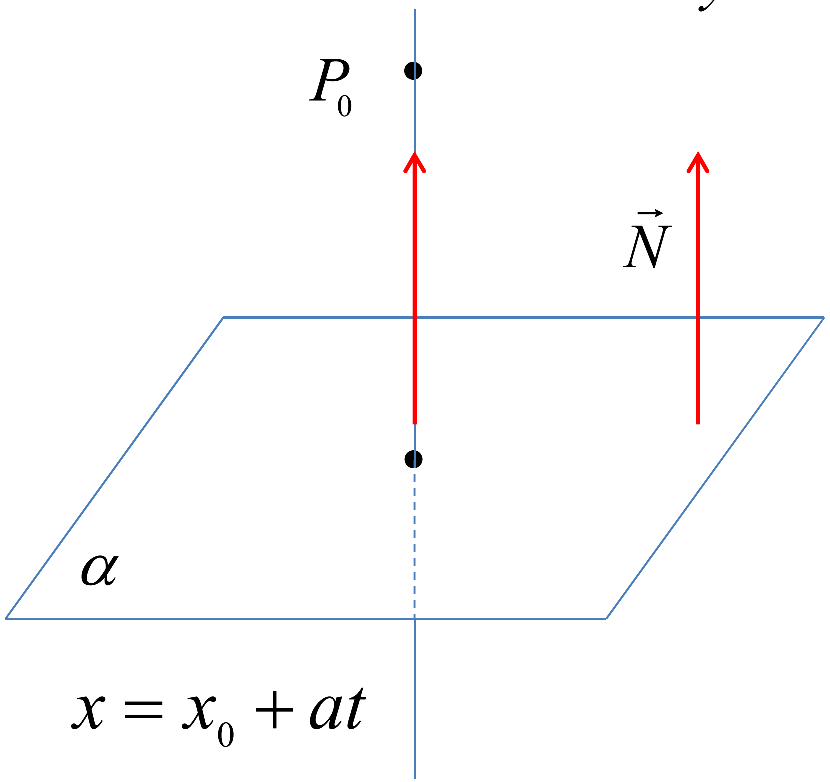
Kanonske jednačine

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

$$\frac{x+1}{-2}=\frac{y-2}{-3}=\frac{z+3}{2}$$

PRIMER*

Napisati parametarske jednačine prave koja sadrži tačku $P_0(-1,2,-3)$
i normalna je na ravan $-2x-3y+2z+4=0$.



$$x = x_0 + at$$
$$y = y_0 + bt \qquad t \in R$$
$$z = z_0 + ct$$

$p:$

$P_0(-1,2,-3)$

$\vec{p} = \vec{N} = (-2,-3,2)$

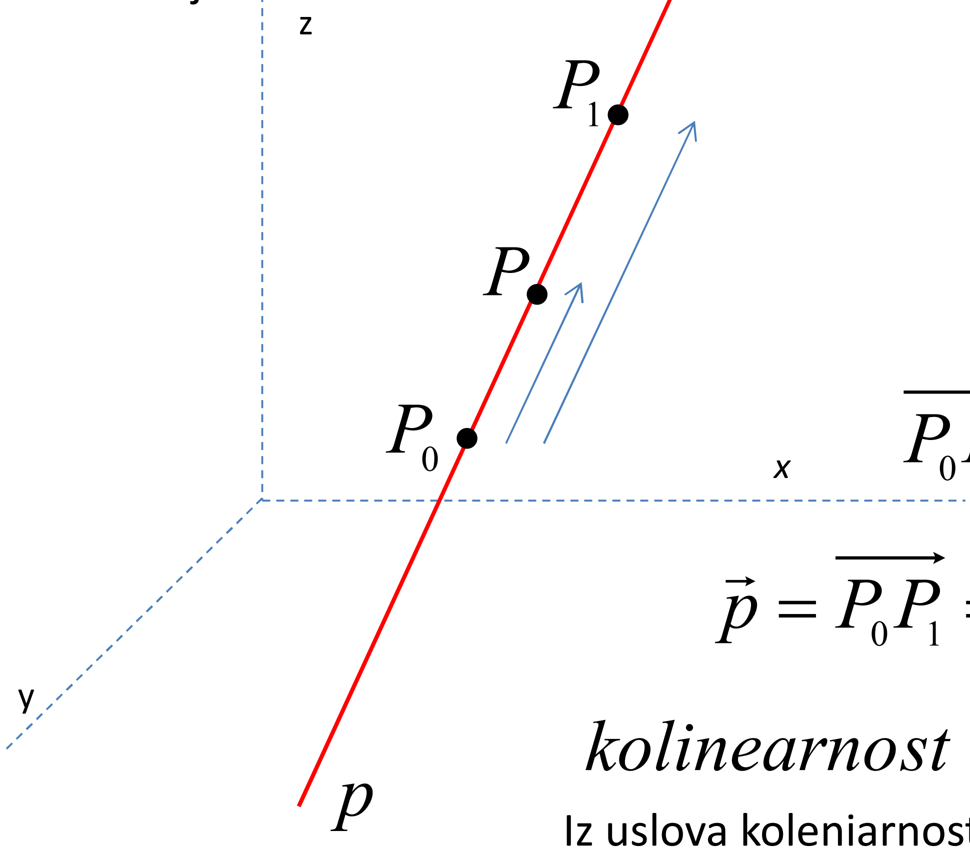
$$\frac{x+1}{-2} = \frac{y-2}{-3} = \frac{z+3}{2} = t$$

Parametarske jednačine

$$x = -1 - 2t$$
$$y = 2 - 3t \qquad t \in R$$
$$z = -3 + 2t$$

JEDNAČINE PRAVE

Prava odredjena sa dve tačke



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

$$\vec{p} = \overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

$$kolinearnost \ \ \overrightarrow{P_0P} \ i \ \overrightarrow{P_0P_1}$$

Iz uslova koleniarnosti slede jednačine prave:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

PRIMER

Napisati jednačine prave odredjene sa dve tacke $P_0(1,-2,3)$ i $P_1(3,4,1)$.

$$\vec{p} = \overrightarrow{P_0P_1} = (3-1, 4+2, 1-3) = (2,6,-2)$$

$$\vec{p} = (2,6,-2)$$

$p:$ $P_0(1,-2,3)$

$\vec{p} = (2,6,-2)$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\frac{x-1}{2} = \frac{y+2}{6} = \frac{z-3}{-2} \quad \Big| \cdot 2 \quad \Rightarrow \quad x-1 = \frac{y+2}{3} = -(z-3)$$

PRIMER*

Napisati jednačine prave odredjene sa dve tacke $P_0(1,-2,3)$ i $P_1(3,4,1)$.

$$\vec{p}=\overrightarrow{P_0P_1}=(3-1,4+2,1-3)=(2,6,-2)$$

$$\vec{p}=(2,6,-2) \qquad \vec{p}=(1,3,-1)$$

$p:$ $P_0(1,-2,3)$

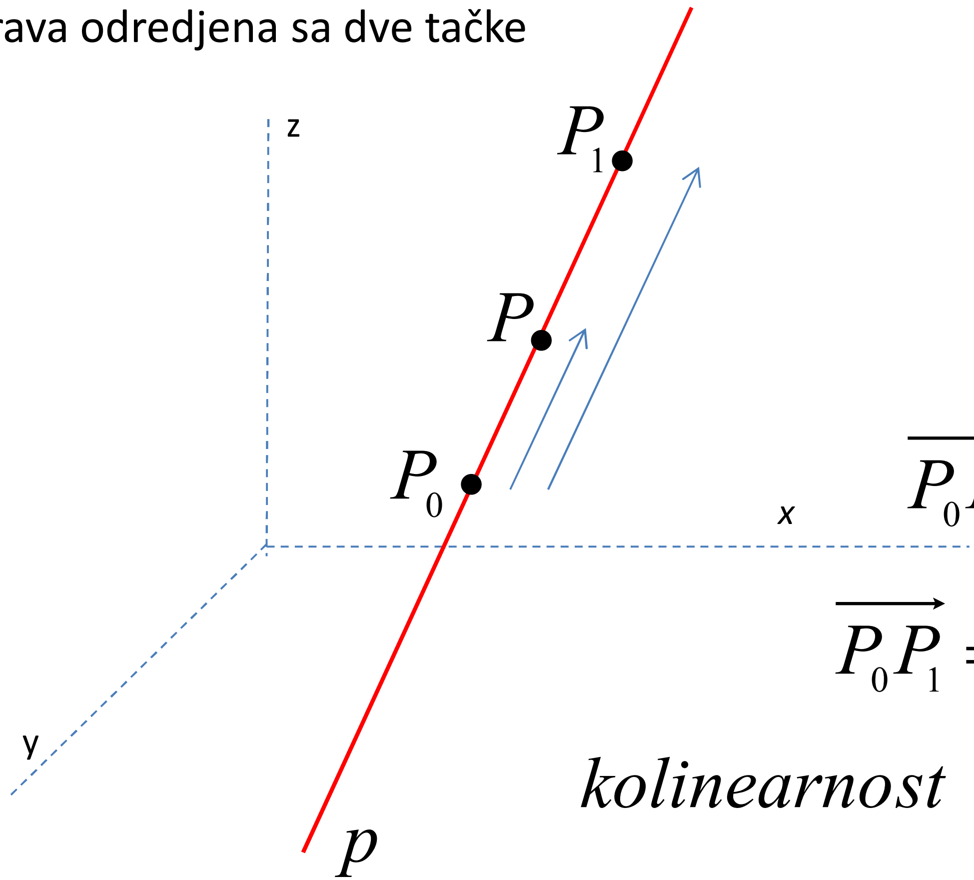
$\vec{p}=(1,3,-1)$

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

$$\frac{x-1}{1}=\frac{y+2}{3}=\frac{z-3}{-1} \qquad x-1=\frac{y+2}{3}=-(z-3)$$

PARAMETARSKE JEDNAČINE PRAVE

Prava odredjena sa dve tačke



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

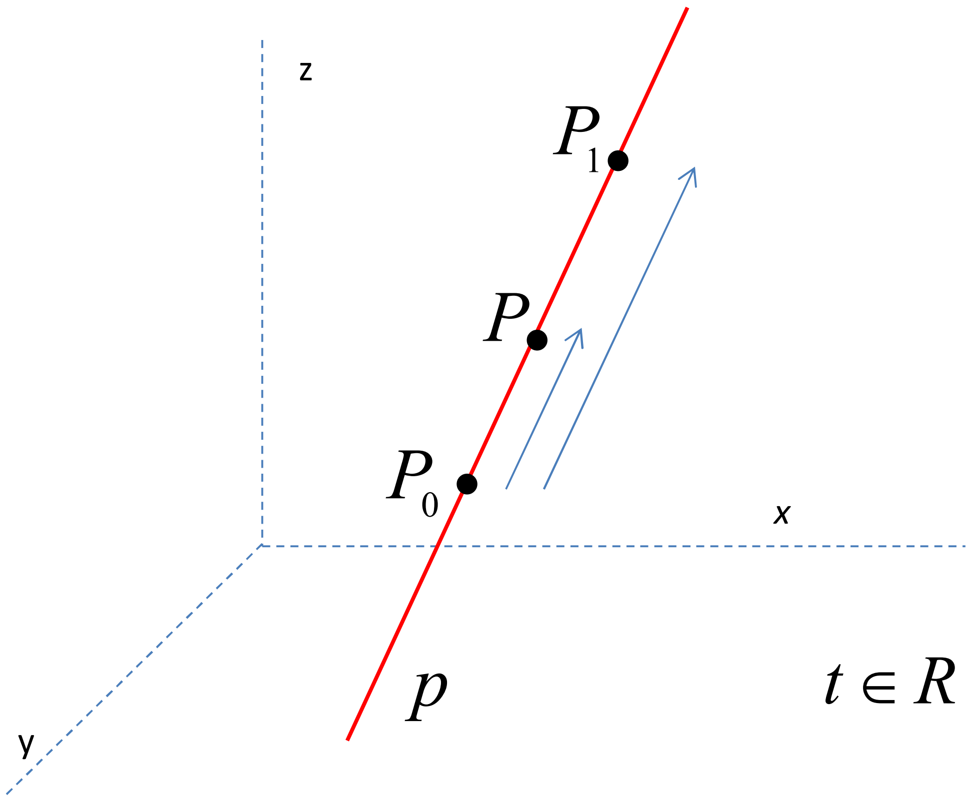
$$\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

$$kolinearnost\ \ \overrightarrow{P_0P}\ i\ \vec{p} \Leftrightarrow \overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1}$$

$$(x - x_0, y - y_0, z - z_0) = t(x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

$$x - x_0 = t(x_1 - x_0),\ y - y_0 = t(y_1 - y_0),\ \ z - z_0 = t(z_1 - z_0)$$

PARAMETARSKE JEDNAČINE PRAVE



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$x - x_0 = t\,(x_1 - x_0)$$
$$y - y_0 = t\,(y_1 - y_0) \qquad t \in R$$
$$z - z_0 = t\,(z_1 - z_0)$$

$$x = (1-t)x_0 + t\,x_1$$
$$y = (1-t)y_0 + t\,y_1 \qquad t \in R$$
$$z = (1-t)z_0 + t\,z_1$$

$$x = x_0 + t\,(x_1 - x_0)$$
$$y = y_0 + t\,(y_1 - y_0) \qquad t \in R$$
$$z = z_0 + t\,(z_1 - z_0)$$

PRIMER

Napisati parametarske jednačine prave odredjene sa dve tacke

$P_0(1,-2,3)$ i $P_1(3,4,1)$.

$p:$

$P_0(x_0, y_0, z_0)$

$P_1(x_1, y_1, z_1)$

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0)$$
$$z = z_0 + t(z_1 - z_0)$$

$t \in R$

$p:\ P_0(1,-2,3)\quad P_1(3,4,1)$

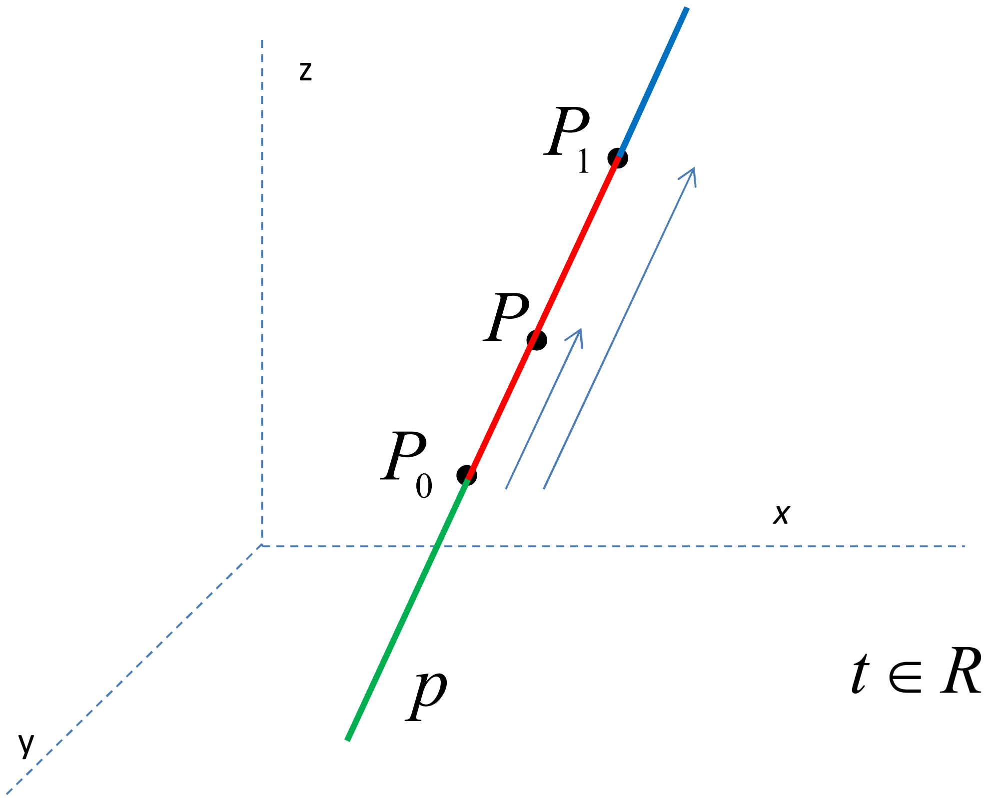
$$x = 1 + t(3-1)$$
$$y = -2 + t(4+2)$$
$$z = 3 + t(1-3)$$

$t \in R$

$$x = 1 + 2t$$
$$y = -2 + 6t$$
$$z = 3 - 2t$$

$t \in R$

PARAMETARSKE JEDNAČINE PRAVE



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in p$$

$$t \in R$$

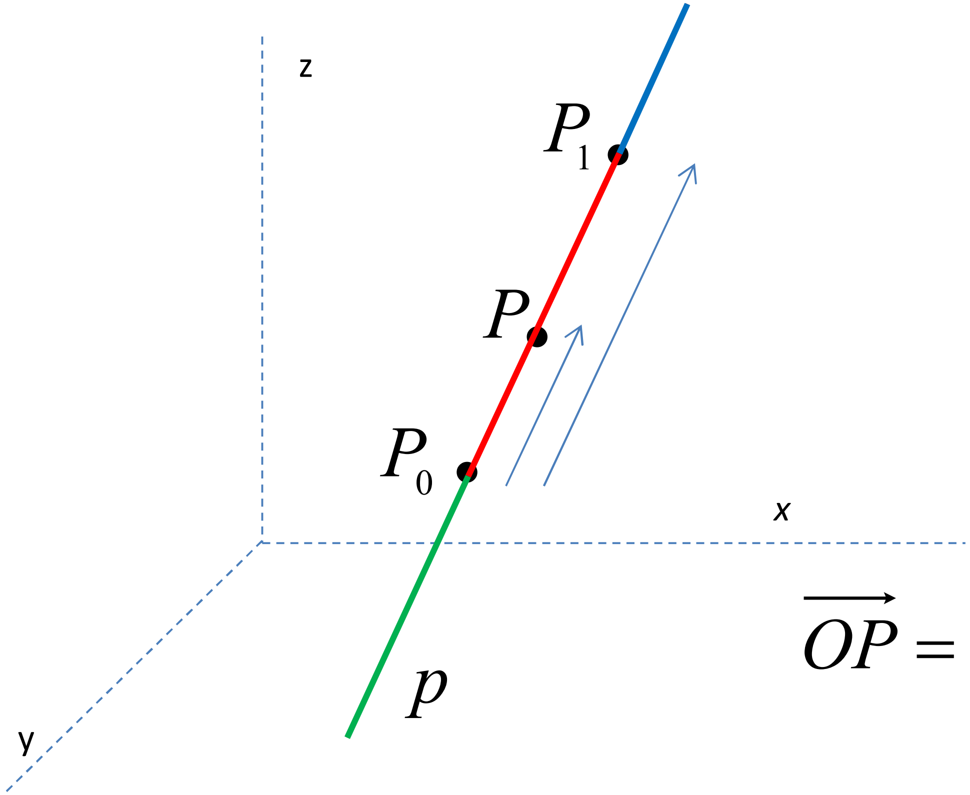
$$x = x_0 + t(x_1 - x_0)$$

$$y = y_0 + t(y_1 - y_0)$$

$$z = z_0 + t(z_1 - z_0)$$

$t < 0$ | $t = 0$, P_0 | $0 < t < 1$ | $t = 1$, P_1 | $t > 1$

PARAMETARSKE JEDNAČINE PRAVE



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

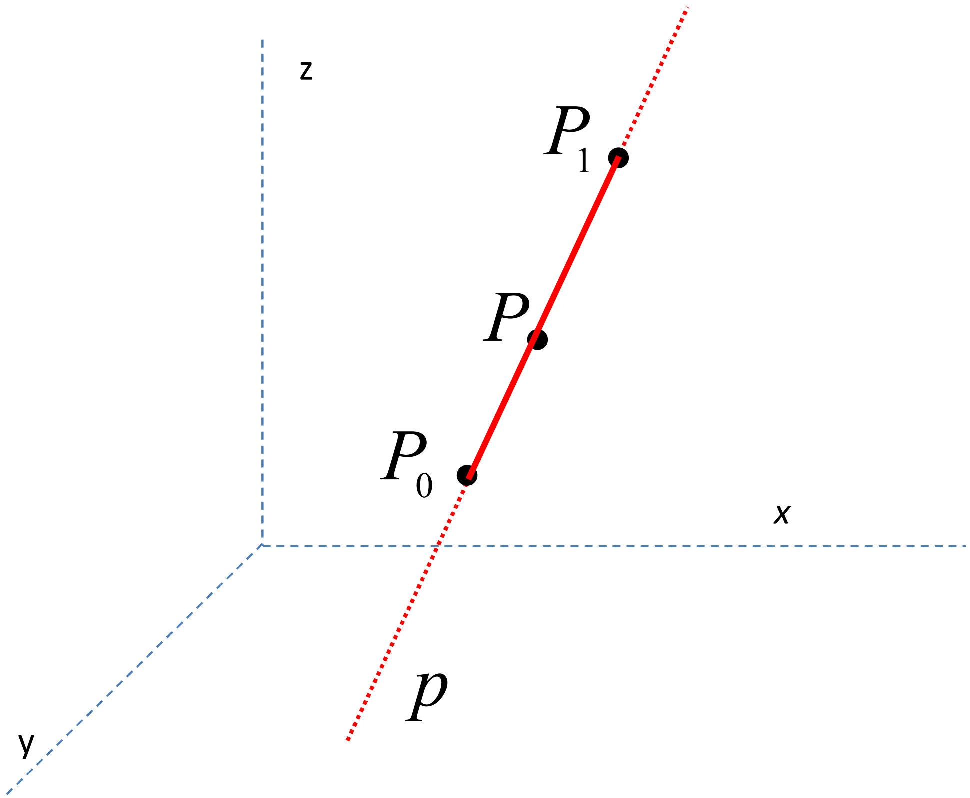
$$P(x, y, z) \in p$$

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1}$$

$$\overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1}$$

$t < 0$	$t = 0$	$0 < t < 1$	$t = 1$	$t > 1$
	P_0		P_1	

PARAMETARSKE JEDNAČINE DUŽI



$p:$

$P_0(x_0, y_0, z_0)$

$P_1(x_1, y_1, z_1)$

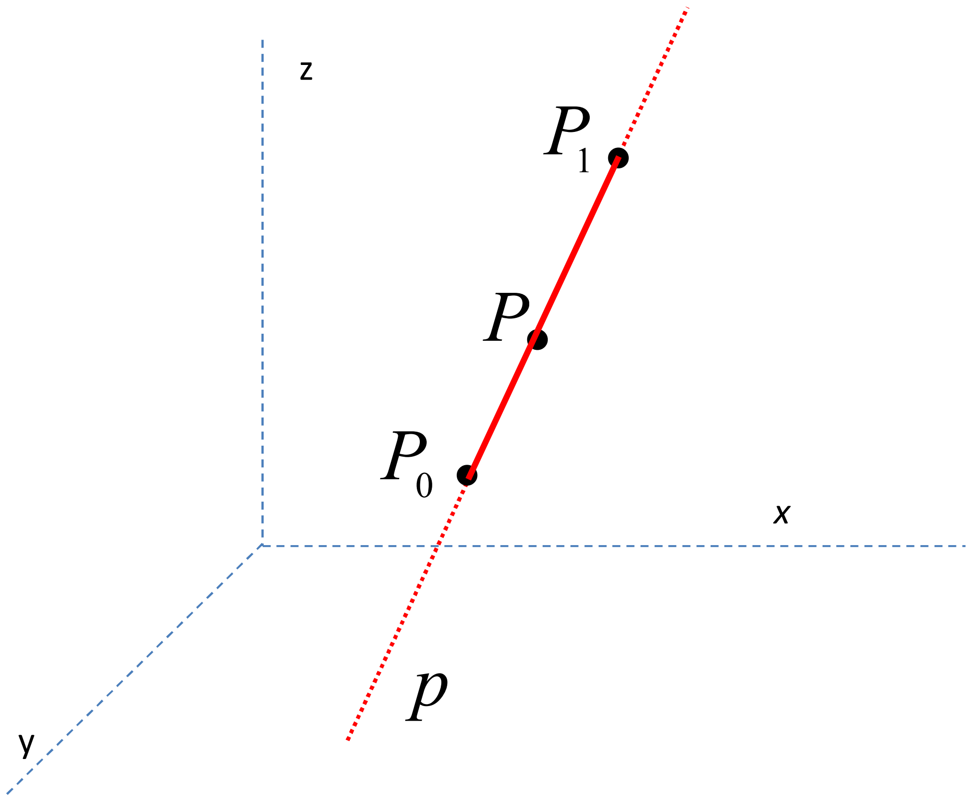
$P(x, y, z) \in P_0P_1$

Jednačine duži P_0P_1

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0)$$
$$z = z_0 + t(z_1 - z_0)$$

$$0 \leq t \leq 1$$

PARAMETARSKE JEDNAČINE DUŽI



$$p:$$

$$P_0(x_0, y_0, z_0)$$

$$P_1(x_1, y_1, z_1)$$

$$P(x, y, z) \in P_0P_1$$

Jednačine duži P_0P_1

$$x = (1-t)x_0 + t\,x_1$$

$$y = (1-t)y_0 + t\,y_1$$

$$z = (1-t)z_0 + t\,z_1$$

$$0 \le t \le 1$$

PARAMETARSKE JEDNAČINE PRAVE

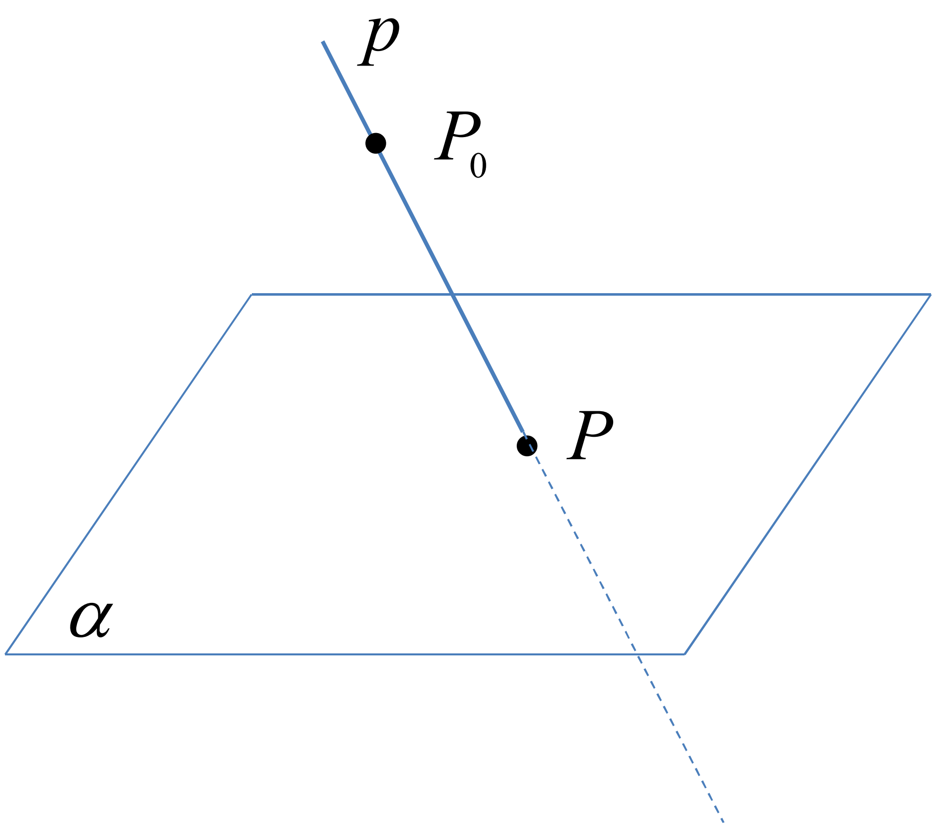
Prelazak sa kanonskog na parametarski oblik

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}=t \qquad t\in R$$

$$\begin{aligned} x&=x_0+at \\ y&=y_0+bt \qquad t\in R \\ z&=z_0+ct \end{aligned}$$

PRODOR PRAVE KROZ RAVAN



$$p: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\alpha: \quad Ax + By + Cz + D = 0$$

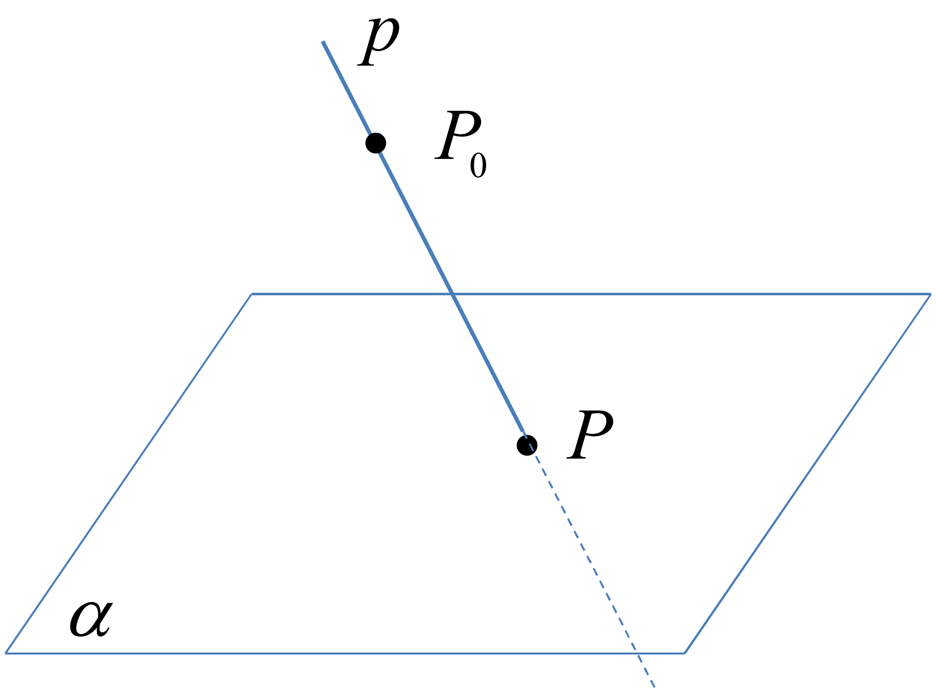
$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

$$x = at + x_0$$

$$y = bt + y_0 \quad t \in R$$

$$z = ct + z_0$$

PRODOR PRAVE KROZ RAVAN



$$P = p \cap \alpha$$

$$p: \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\alpha: \quad Ax + By + Cz + D = 0$$

$$x = at + x_0$$

$$y = bt + y_0 \qquad t \in R$$

$$z = ct + z_0$$

$$P(x_P, y_P, z_P) = ?$$

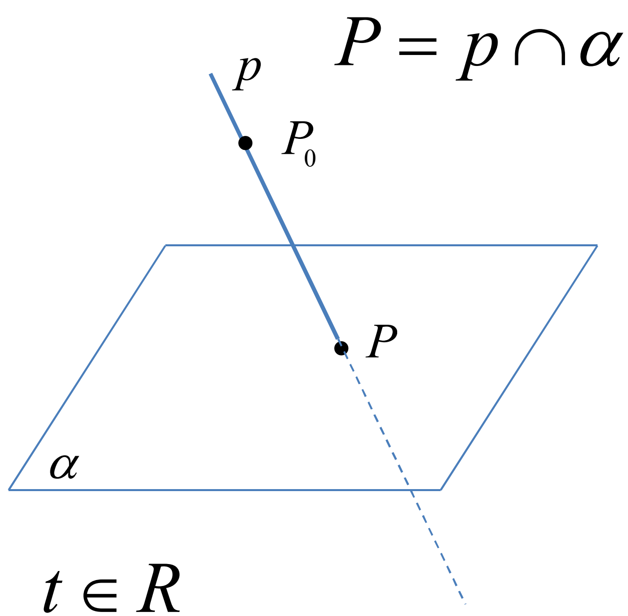
$$A(at + x_0) + B(bt + y_0) + C(ct + z_0) + D = 0 \quad \Rightarrow \quad t = t_P$$

$$x_P = at_P + x_0, \quad y_P = bt_P + y_0, \quad z_P = ct_P + z_0 \quad \Rightarrow \quad P(x_P, y_P, z_P)$$

PRIMER

Odrediti prodor prave $p: \frac{x+1}{-2}=\frac{y-2}{-3}=\frac{z+3}{2}$

kroz ravan $\alpha: \quad -2x-3y+2z+27=0$



$\frac{x+1}{-2}=\frac{y-2}{-3}=\frac{z+3}{2}=t$ Parametrizacija prave

$$x=-1-2t, \quad y=2-3t, \quad z=-3+2t, \quad t\in R$$

$$-2x-3y+2z+27=0$$

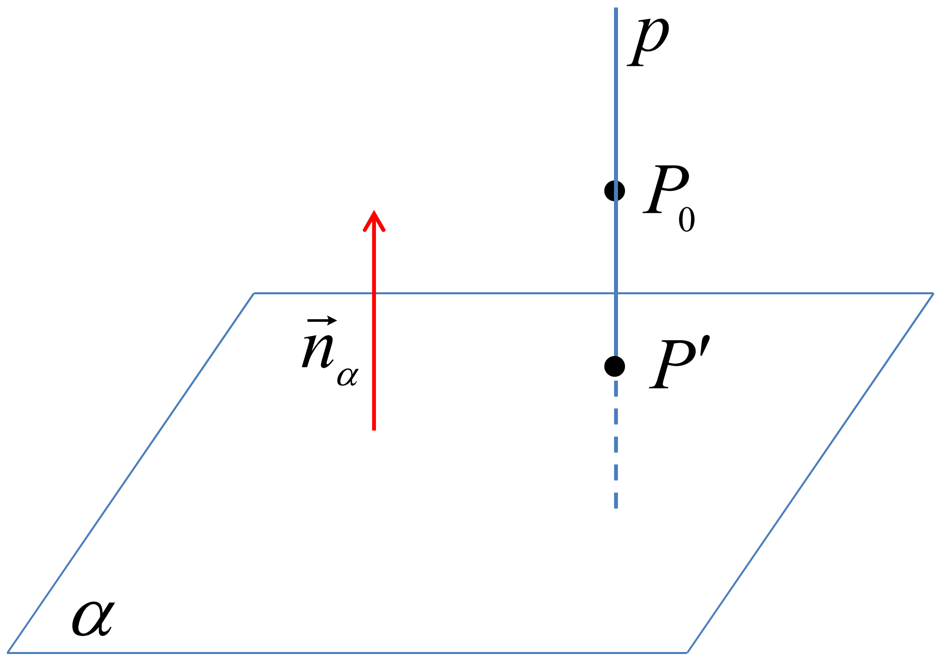
$$-2(-1-2t)-3(2-3t)+2(-3+2t)+27=0$$

$$4t+9t+4t+17=0 \qquad 17t+17=0 \qquad t=-1$$

$$x_P=-1-2\cdot(-1), \quad y_P=2-3\cdot(-1), \quad z_P=-3+2\cdot(-1)$$

$$x_P=1, \quad y_P=5, \quad z_P=-5 \qquad p\cap\alpha=P(1,5,-5)$$

PROJEKCIJA TAČKE NA RAVAN



$$P_0(x_0, y_0, z_0)$$

$$\alpha : Ax + By + Cz + D_1 = 0$$

$$\vec{n}_\alpha = (A, B, C)$$

$$p: \quad (P_0 \in p, \quad p \perp \alpha)$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{p} = \vec{n}_\alpha$$

$$p: \quad \frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

$$p \cap \alpha = P'$$

PRIMER

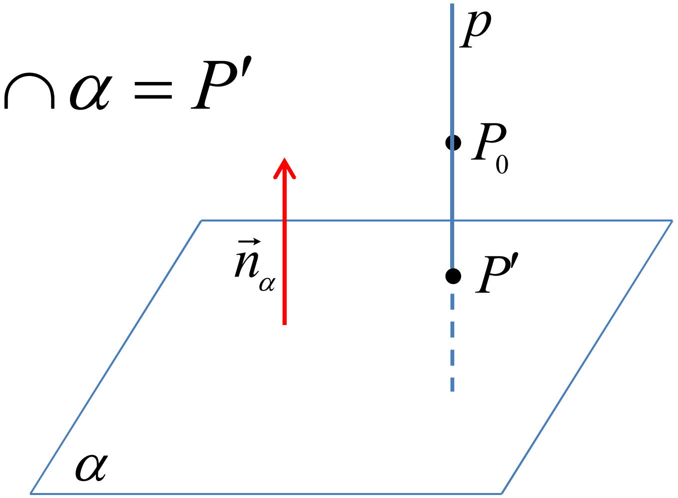
Odrediti ortogonalnu projekciju tačke

$P_0(1,-2,5)$

na ravan

$$\alpha : \quad 2x-3y+z+1=0$$

$$p \cap \alpha = P'$$



$$P_0 \in p, \quad p \perp \alpha$$

$$p : \quad P_0(1,-2,5)$$

$$\vec{p} = \vec{n}_\alpha = (2,-3,1)$$

$$p : \quad \frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-5}{1} = t$$

$$x = 2t+1$$

$$y = -3t-2 \qquad t \in R$$

$$z = t+5$$

$$2(2t+1)-3(-3t-2)+(t+5)+1=0$$

$$14t+14=0 \qquad t=-1$$

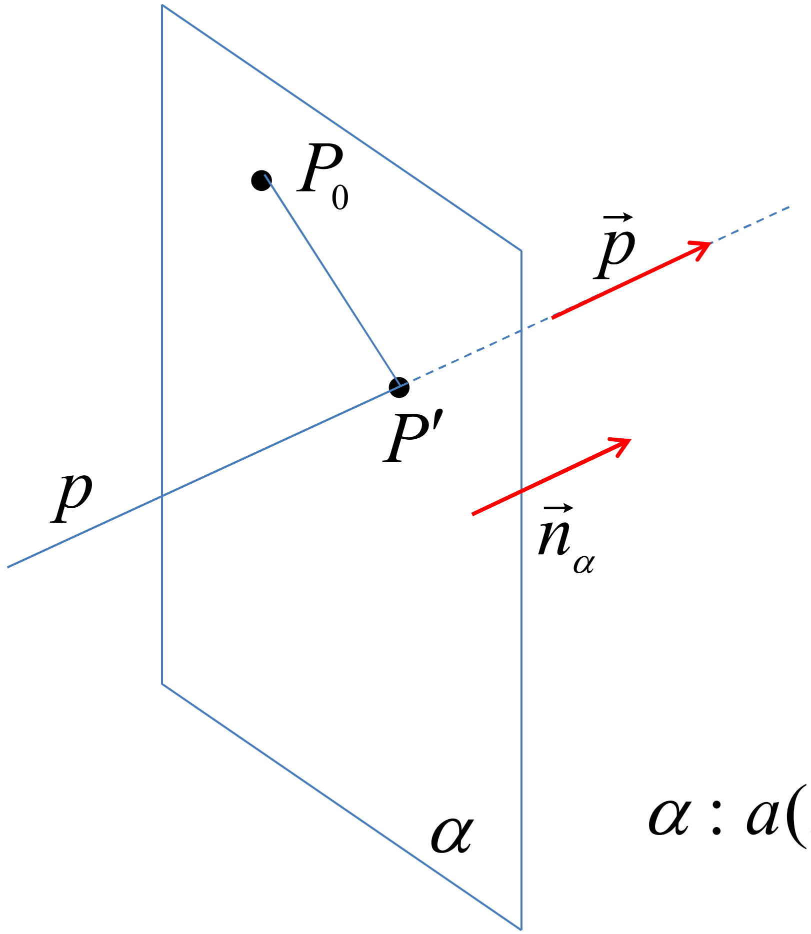
$$x_{P'} = 2\cdot(-1)+1=-1$$

$$y_{P'} = -3\cdot(-1)-2=1$$

$$z_{P'} = -1+5=4$$

$$p \cap \alpha = P'(-1,1,4)$$

PROJEKCIJA TAČKE NA PRAVU



$$P_0(x_0, y_0, z_0)$$

$$p: \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\vec{p} = (a, b, c)$$

$$\alpha: \quad (P_0 \in \alpha, \quad \alpha \perp p)$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{n}_\alpha = \vec{p}$$

$$\vec{n}_\alpha = (a, b, c)$$

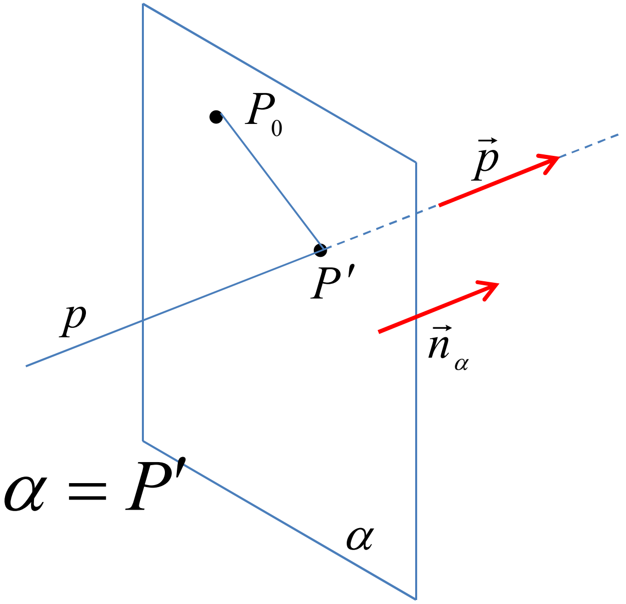
$$\alpha : a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$p \cap \alpha = P'$$

PRIMER

Odrediti ortogonalnu projekciju tačke $P_0(2,\ 2,\ 1)$

na pravu $p:\quad \dfrac{x-1}{2}=\dfrac{y+2}{-3}=\dfrac{z-5}{1}$



$\alpha:\qquad (P_0\in\alpha,\quad \alpha\perp p)$

$P_0(2,2,1)$

$\vec{n}_\alpha=\vec{p}=(2,-3,\ 1)$

$p\cap\alpha=P'$

$$\alpha: 2(x-2)-3(y-2)+(z-1)=0$$

$$2x-3y+z+1=0$$

$$p:\ \ x=2t+1,\ \ y=-3t-2,\ \ z=t+5$$

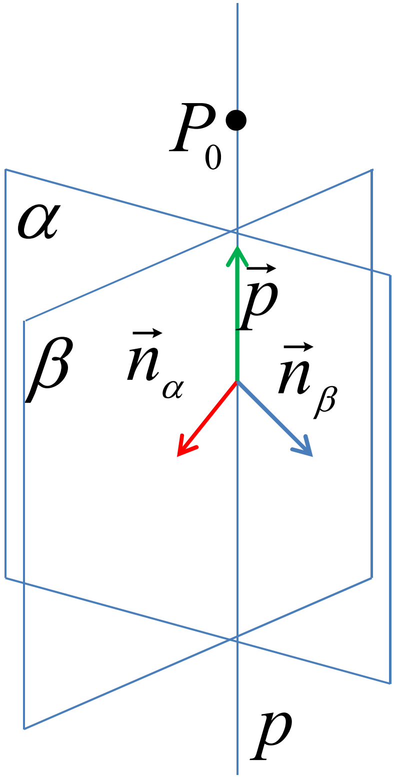
$$2(2t+1)-3(-3t-2)+(t+5)+1=0$$

$$14t+14=0\Rightarrow\ t=-1$$

$$x_{P'}=-1,\ \ y_{P'}=1,\ \ z_{P'}=4$$

$$p\cap\alpha=P'(-1,1,\ 4)$$

PRAVA KAO PRESEK DVE RAVNI



$$\alpha : \quad A_1x + B_1y + C_1z + D_1 = 0$$
$$\beta : \quad A_2x + B_2y + C_2z + D_2 = 0$$

$$p = \alpha \cap \beta$$

$$\vec{n}_\alpha = (A_1, B_1, C_1) \quad \vec{n}_\beta = (A_2, B_2, C_2)$$

$$\vec{n}_\alpha \perp \alpha \Rightarrow \vec{n}_\alpha \perp p$$

$$\vec{n}_\beta \perp \beta \Rightarrow \vec{n}_\beta \perp p$$

$$\vec{p} \perp \vec{n}_\alpha, \ \vec{p} \perp \vec{n}_\beta \quad \Rightarrow \quad \vec{p} = \vec{n}_\alpha \times \vec{n}_\beta$$

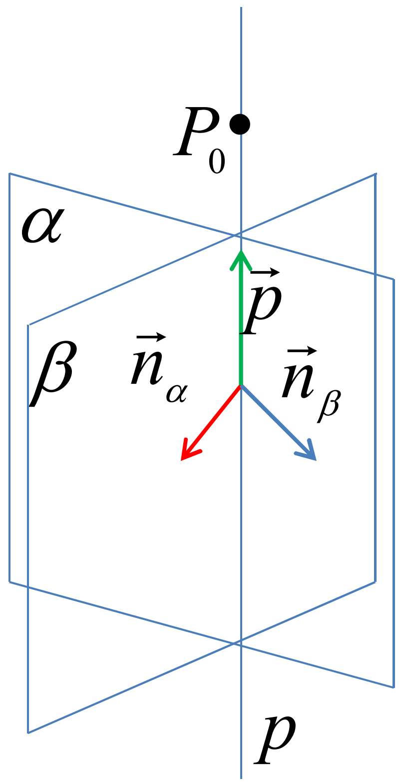
$p:$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (a, b, c)$$

$$P_0(x_0, y_0, z_0) \in p$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

PRAVA KAO PRESEK DVE RAVNI



$$\alpha: \quad A_1x + B_1y + C_1z + D_1 = 0$$

$$\beta: \quad A_2x + B_2y + C_2z + D_2 = 0$$

$$p = \alpha \cap \beta$$

$$p: \quad \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$\vec{n}_\alpha = (A_1, B_1, C_1) \quad \vec{n}_\beta = (A_2, B_2, C_2)$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (a, b, c)$$

Kanonski oblik:

$$p: \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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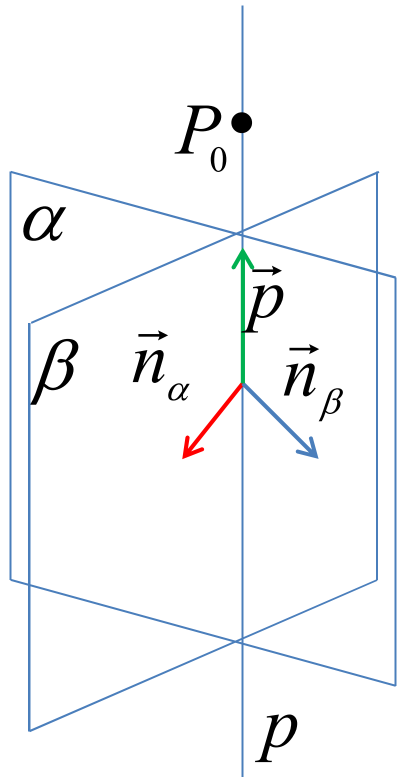
Napisati kanonske jednačine prave

$$p: \quad \begin{cases} 2x - 3y + z + 1 = 0 \\ -x + y - 3 = 0 \end{cases}$$

zadate kao presek dve ravni.



$$\vec{n}_\alpha = (2,-3,1) \qquad \vec{n}_\beta = (-1,1,0)$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & 1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} = -\vec{i} - \vec{j} - \vec{k}$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (-1,-1,-1)$$

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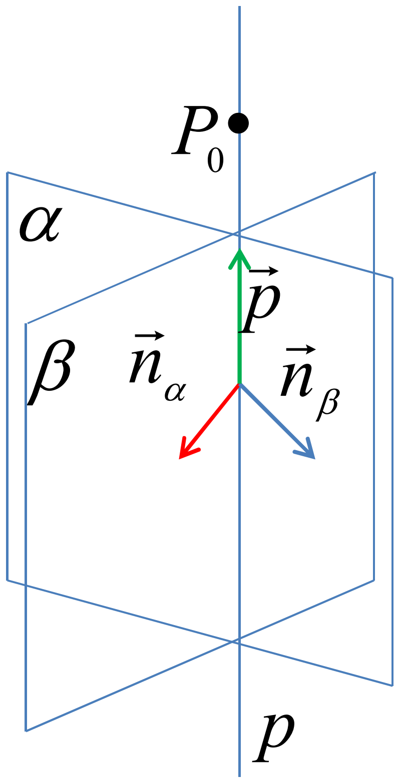
Napisati kanonske jednačine prave

$$p: \quad \begin{cases} 2x-3y+z+1=0 \\ -x+y-3=0 \end{cases}$$

zadate kao presek dve ravni.



$$\vec{n}_\alpha = (2,-3,1) \qquad \vec{n}_\beta = (-1,1,0)$$

$$\vec{p} = \vec{n}_\alpha \times \vec{n}_\beta = (-1,-1,-1)$$

$$P_0(x_0, y_0, z_0) \in p$$

$$2x-3y+z+1=0$$
$$-x+y-3=0$$

$$z=0$$

$$\left.\begin{aligned} 2x-3y+1&=0 \\ -x+y-3&=0 \;\Big|\cdot 2 \end{aligned}\right\} +$$

$$-y-5=0 \Rightarrow y=-5$$

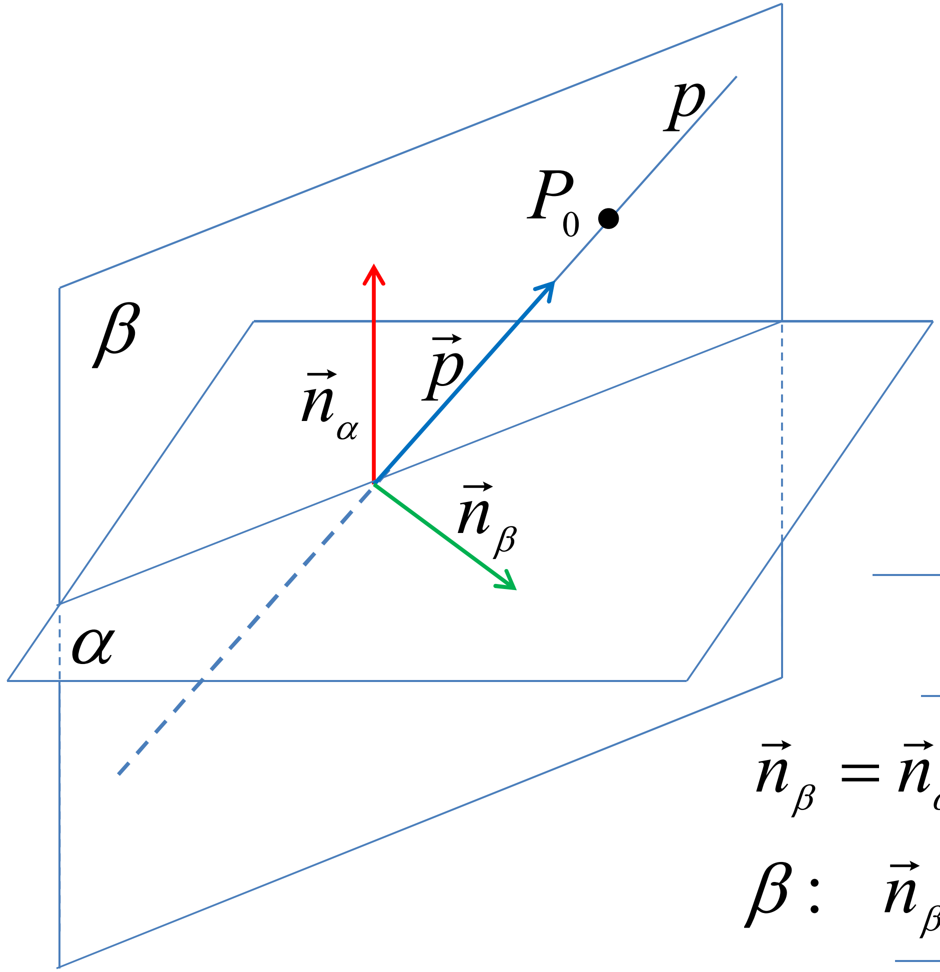
$$-x-5-3=0 \Rightarrow x=8$$

$$p: \quad P_0(8,-5,0)$$
$$\vec{p} = (-1,-1,-1)$$

$$p: \quad \frac{x-8}{-1} = \frac{y+5}{-1} = \frac{z}{-1} \quad \Big| \cdot (-1)$$

$$p: \quad x-8 = y+5 = z$$

PROJEKCIJA PRAVE NA RAVAN



$$\alpha: \quad A_1 x + B_1 y + C_1 z + D_1 = 0$$

$$p: \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\vec{n}_\alpha = (A_1, B_1, C_1)$$

$$\vec{p} = (a, b, c)$$

$$\beta \supset p, \quad \beta \perp \alpha$$

$$\vec{n}_\beta = \vec{n}_\alpha \times \vec{p}, \quad P_0(x_0, y_0, z_0) \in p \subset \beta$$

$$\beta: \quad \vec{n}_\beta = \vec{n}_\alpha \times \vec{p}, \quad P_0(x_0, y_0, z_0) \in \beta$$

$$\alpha \cap \beta = p'$$

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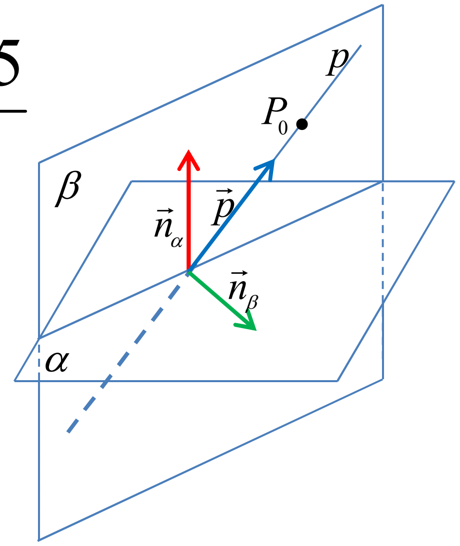
Odrediti projekciju prave $p: \quad \dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z-5}{1}$

na ravan $\alpha: \quad x - 2z + 4 = 0$.



$\vec{p} = (2, -3, 1) \quad \vec{n}_\alpha = (1, 0, -2)$

$\beta: \quad \vec{n}_\beta = \vec{n}_\alpha \times \vec{p} \quad P_0(1, -2, 5) \in p \subset \beta$

$$\vec{n}_\beta = \vec{n}_\alpha \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 0 & -2 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & 1 \\ 0 & -2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 6\vec{i} + 5\vec{j} + 3\vec{k}$$

$\beta: \quad 6(x-1) + 5(y+2) + 3(z-5) = 0$

$\beta: \quad 6x + 5y + 3z - 11 = 0$

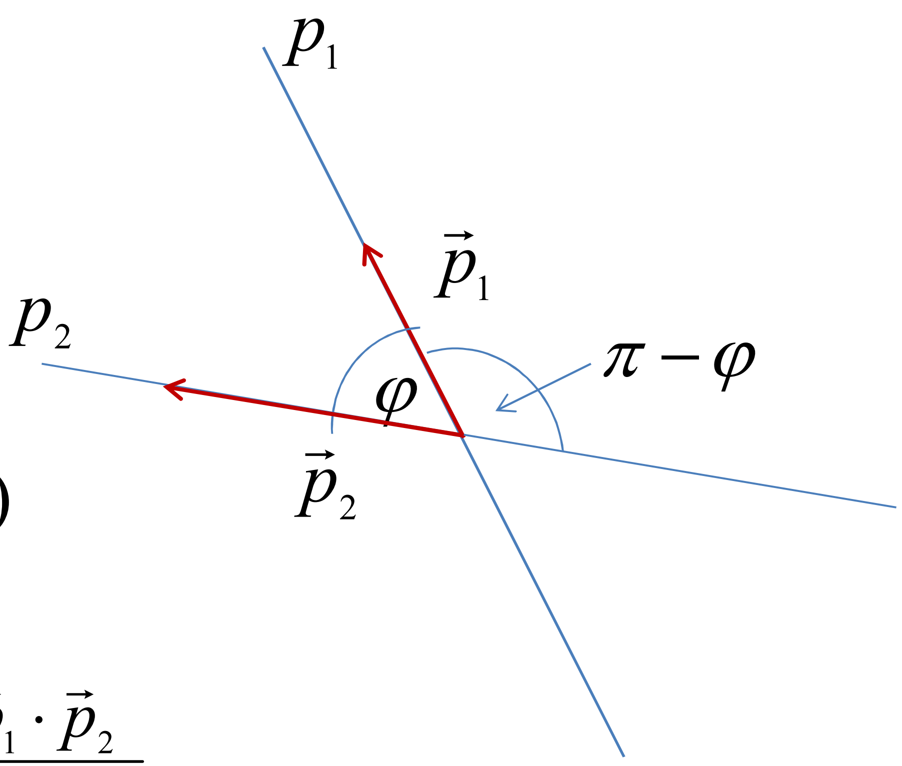
$p' = \alpha \cap \beta$

$$p': \begin{cases} x - 2z + 4 = 0 \\ 6x + 5y + 3z - 11 = 0 \end{cases}$$

UGAO IZMEDJU DVE PRAVE

$$\vec{p}_1 \cdot \vec{p}_2 = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

Ugao izmedju dve prave je ugao za koji treba zarotirati jednu pravu da bi se poklopila sa drugom pravom.



$$p_1: \quad \frac{x - x_0}{a_1} = \frac{y - y_0}{b_1} = \frac{z - z_0}{c_1}$$

$$p_2: \quad \frac{x - x_0}{a_2} = \frac{y - y_0}{b_2} = \frac{z - z_0}{c_2}$$

$$\vec{p}_1 = (a_1, b_1, c_1) \qquad \vec{p}_2 = (a_2, b_2, c_2)$$

$$\vec{p}_1 \cdot \vec{p}_2 = |\vec{p}_1| \cdot |\vec{p}_2| \cdot \cos\varphi$$

$$\cos\varphi = \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| \cdot |\vec{p}_2|} \qquad \varphi = arc\cos\frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| \cdot |\vec{p}_2|}$$

$\varphi, \quad \pi - \varphi$ su dva ugla od kojih je jedan ostar a drugi tup

PRIMER

Odrediti ugao izmedju pravih

$$p_1: \quad \frac{x-2}{1} = \frac{y-3}{3} = \frac{z+1}{-2}$$

$$p_2: \quad x-4 = -(x+1) = -z$$

$$\vec{p}_1 = (1,3,-2)$$

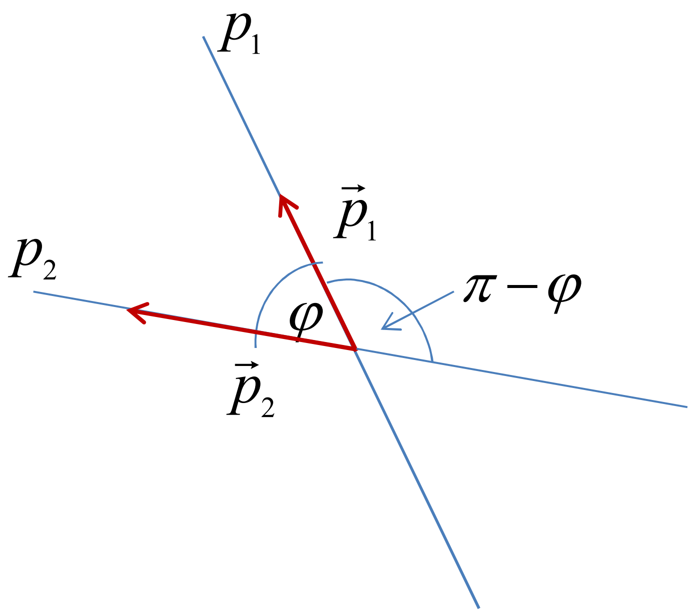
$$\vec{p}_2 = (1,-1,-1)$$

$$\vec{p}_1 \cdot \vec{p}_2 = 1 \cdot 1 + 3 \cdot (-1) + (-2) \cdot (-1) = 0$$

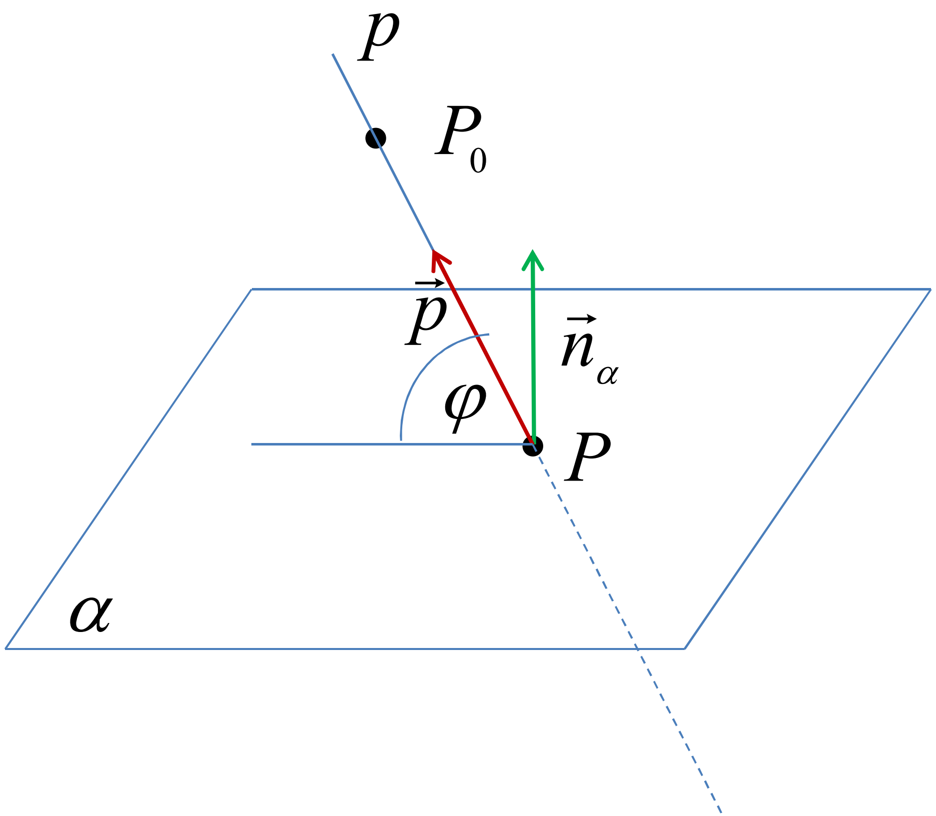
$$\cos\varphi = \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| \cdot |\vec{p}_2|} = \frac{0}{|\vec{p}_1| \cdot |\vec{p}_2|} = 0$$

$$\varphi = arc\cos 0 = \frac{\pi}{2}$$

$$\pi - \varphi = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$



UGAO IZMEDJU PRAVE I RAVNI



$$p:\ \frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

$$\alpha:\quad Ax+By+Cz+D=0$$

$$\vec{p}=(a,b,c)$$

$$\vec{n}_\alpha=(A,B,C)$$

$$\angle(\vec{p},\vec{n}_\alpha)=\frac{\pi}{2}-\varphi$$

$$\cos\left(\frac{\pi}{2}-\varphi\right)=\frac{\vec{p}\cdot\vec{n}}{|\vec{p}||\vec{n}_\alpha|}$$

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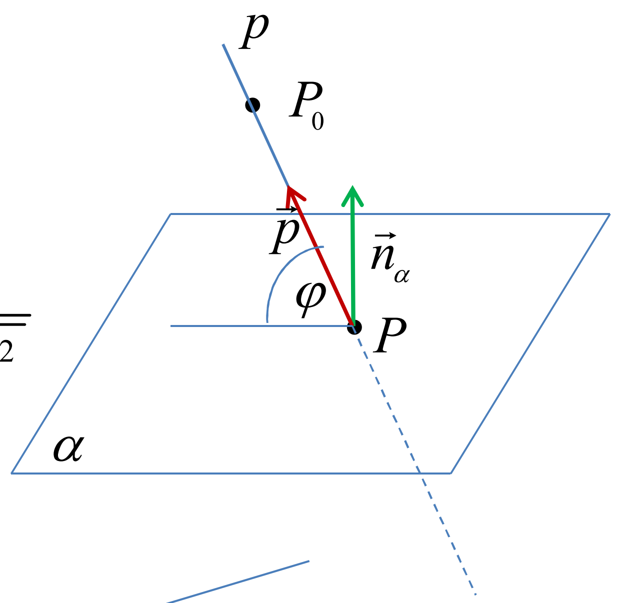
Odrediti ugao izmedju prave

$$p: \quad \frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{2}$$

i ravni $\quad \alpha: \ 2x - y + 1 = 0$

$$\vec{p} = (1,2,2)$$

$$\vec{n}_\alpha = (2,-1,0)$$

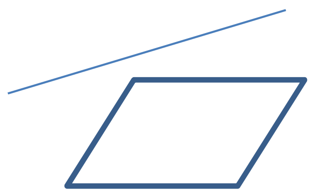


$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{\vec{p}\cdot\vec{n}}{|\vec{p}||\vec{n}_\alpha|}$$

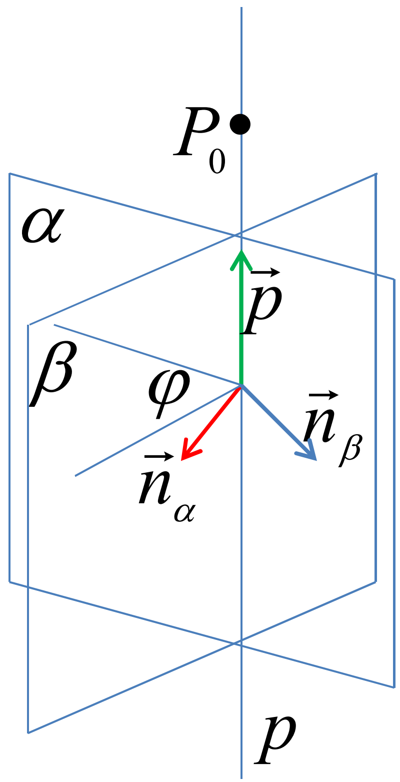
$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{1\cdot 2 + 2\cdot(-1) + 2\cdot 0}{\sqrt{1^2+2^2+2^2}\cdot\sqrt{2^2+(-1)^2+0^2}}$$

$$\cos\left(\frac{\pi}{2} - \varphi\right) = 0$$

$$\frac{\pi}{2} - \varphi = \frac{\pi}{2} \ \Rightarrow \ \varphi = 0 \ \Rightarrow p \,||\, \alpha$$



UGAO IZMEDJU DVE RAVNI



Ugao izmedju dve ravni

$$\alpha: \quad A_1x + B_1\,y + C_1z + D_1 = 0$$

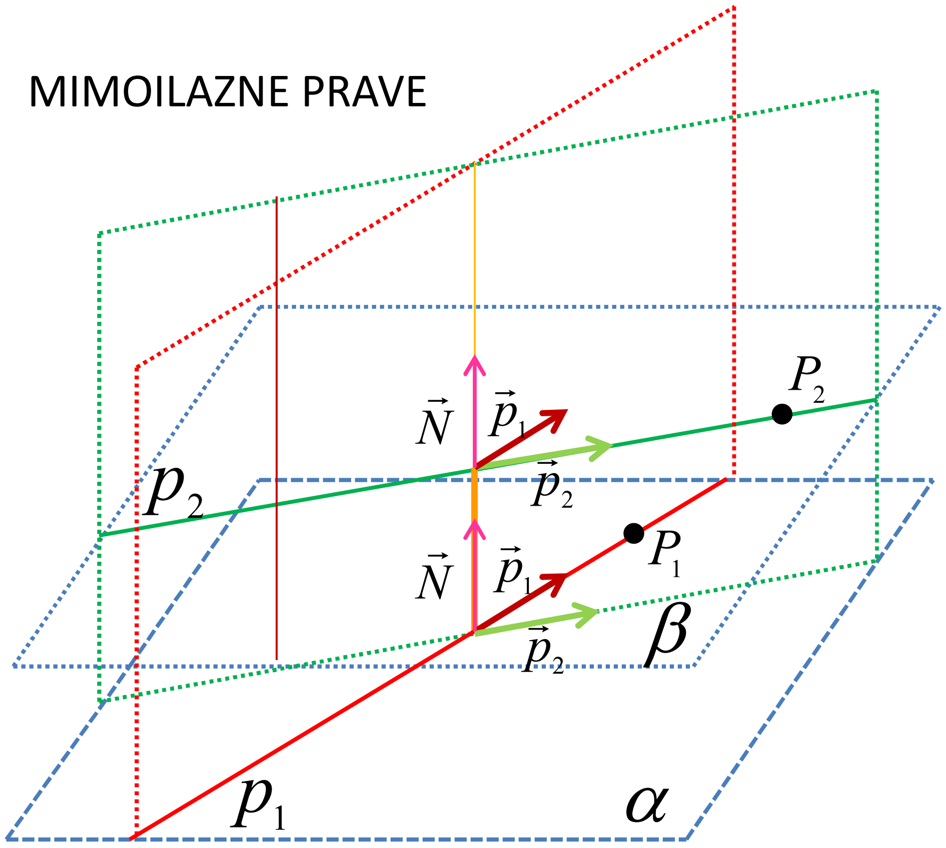
$$\beta: \quad A_2x + B_2\,y + C_2z + D_2 = 0$$

$$\vec{n}_\alpha = (A_1, B_1, C_1)$$

$$\vec{n}_\beta = (A_2, B_2, C_2)$$

$$\cos\varphi = \frac{\vec{n}_\alpha \cdot \vec{n}_\beta}{|\vec{n}_\alpha||\vec{n}_\beta|}$$

MIMOILAZNE PRAVE



$$p_1: \quad \frac{x-x_0}{a_1}=\frac{y-y_0}{b_1}=\frac{z-z_0}{c_1}$$

$$p_2: \quad \frac{x-x_0}{a_2}=\frac{y-y_0}{b_2}=\frac{z-z_0}{c_2}$$

$$\vec{p}_1=(a_1,b_1,c_1)$$

$$\vec{p}_2=(a_2,b_2,c_2)$$

α Sadrži p_1 i paralelna je p_2.

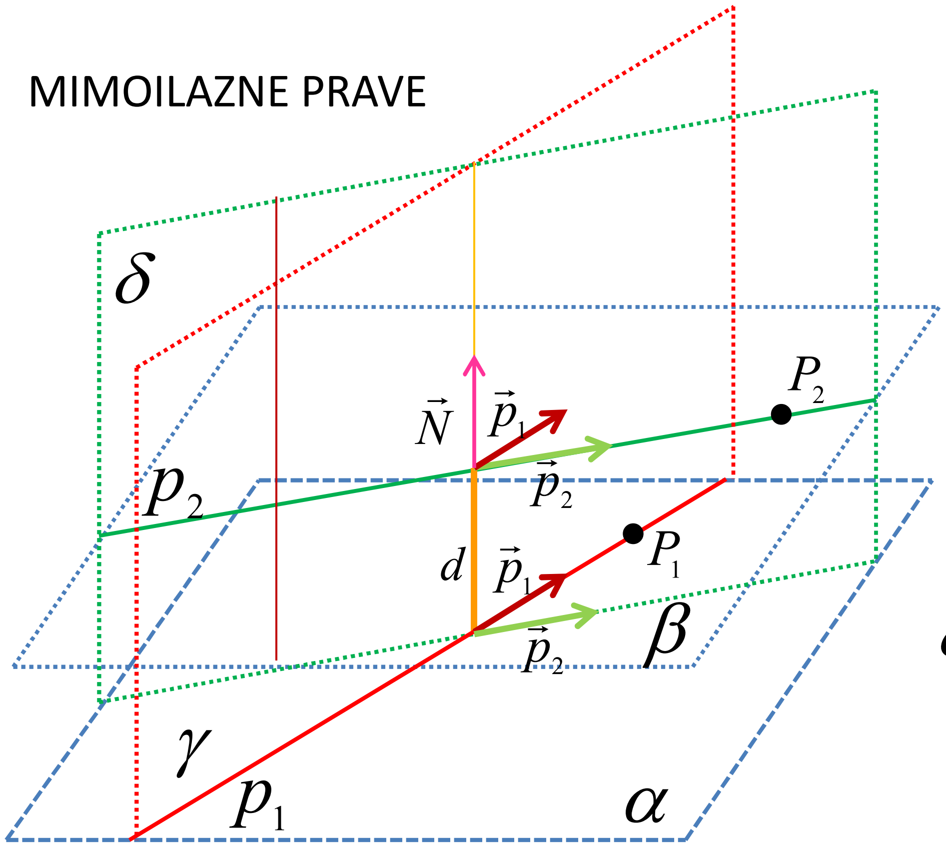
β Sadrži p_2 i paralelna je p_1 .

$$\vec{N}=\vec{p}_1\times\vec{p}_2=(A,B,C)$$

$$\alpha: \vec{N},\ P_1\in p_1 \qquad \alpha: Ax+By+Cz+D_1=0$$

$$\beta: \vec{N},\ P_2\in p_2 \qquad \beta: Ax+By+Cz+D_2=0$$

MIMOILAZNE PRAVE



Najkraće rastojanje izmedju tačaka mimoilazne prave

$$d = \left| \overrightarrow{P_1 P_2} \cdot ort\, \vec{N} \right|$$

$$\gamma : \ \gamma \supseteq p_1, \ \gamma \perp \alpha \ \ (\gamma \perp \beta)$$

$$\delta : \ \delta \supseteq p_2, \ \delta \perp \alpha \ \ (\delta \perp \beta)$$

$$\gamma : \ P_1 \in p_1, \ \vec{n}_\gamma = \vec{p}_1 \times \vec{N}$$

$$\delta : \ P_2 \in p_2, \ \vec{n}_\delta = \vec{p}_2 \times \vec{N}$$

$n = \gamma \cap \delta$ Zajednička normala

$n \cap \alpha = M_1 \in p_1 \quad n \cap \beta = M_2 \in p_2$ najbliže tačke

Rastojanje izmedju tačaka $M_1 \in p_1$ i $M_2 \in p_2$ je najkraće rastojanje izmedju dve mimoilazne prave.

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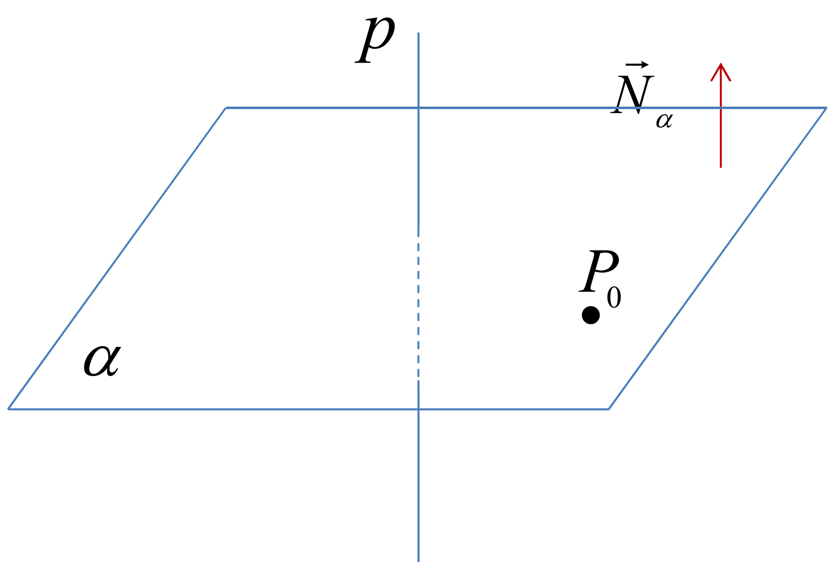
Napisati jednačinu ravni α
koja sadrži tačku

$$P_0(-2,5,3)$$

i normalna je na pravu

$$p: \quad \frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{1}.$$



$$P_0(-2,5,3) \in \alpha \qquad \vec{N}_\alpha = \vec{p} = (1,-1,1)$$

$$\alpha : A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$(x+2) - (y-5) + (z-3) = 0$$

$$x - y + z + 4 = 0$$

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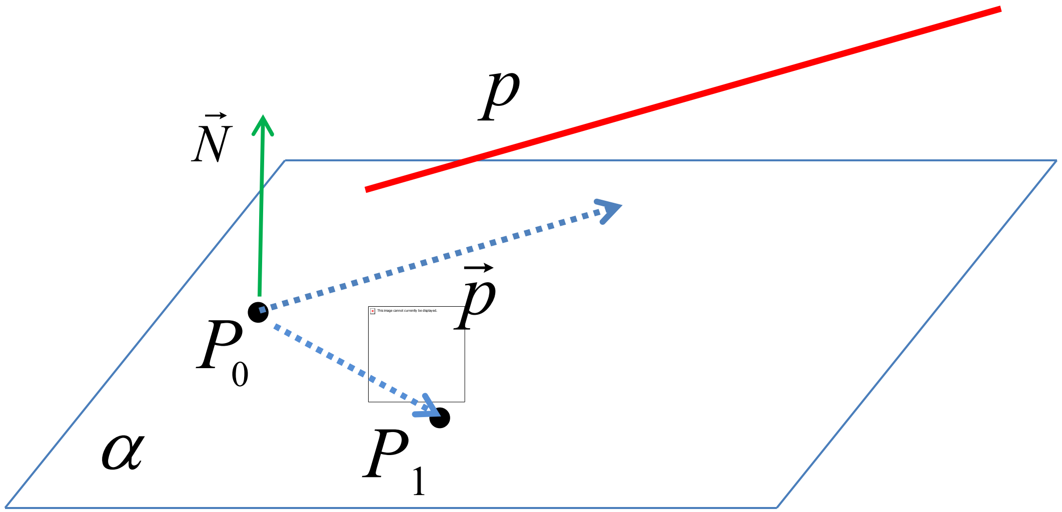
Napisati jednačinu ravni α koja sadrži tačke

$P_0(-2,1,3)$ i $P_1(0,1,-3)$

i paralelna je pravoj

$$\vec{p}\ : \frac{x-1}{1} = \frac{y+1}{2} = -\frac{z}{1}$$



$$\vec{a} = \overrightarrow{P_0P_1} = (0+2, 1-1, -3-3) = (2, 0, -6)$$

$$\vec{b} = \vec{p} = (1,2,-1)$$

$$\vec{N} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -6 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -6 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -6 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 12\vec{i} - 4\vec{j} + 4\vec{k}$$

$$\vec{N} = (12, -4, 4) = (3, -1, 1)$$

$$3(x-0) - (y-1) + (z+3) = 0$$

$$3x - y + z + 4 = 0$$

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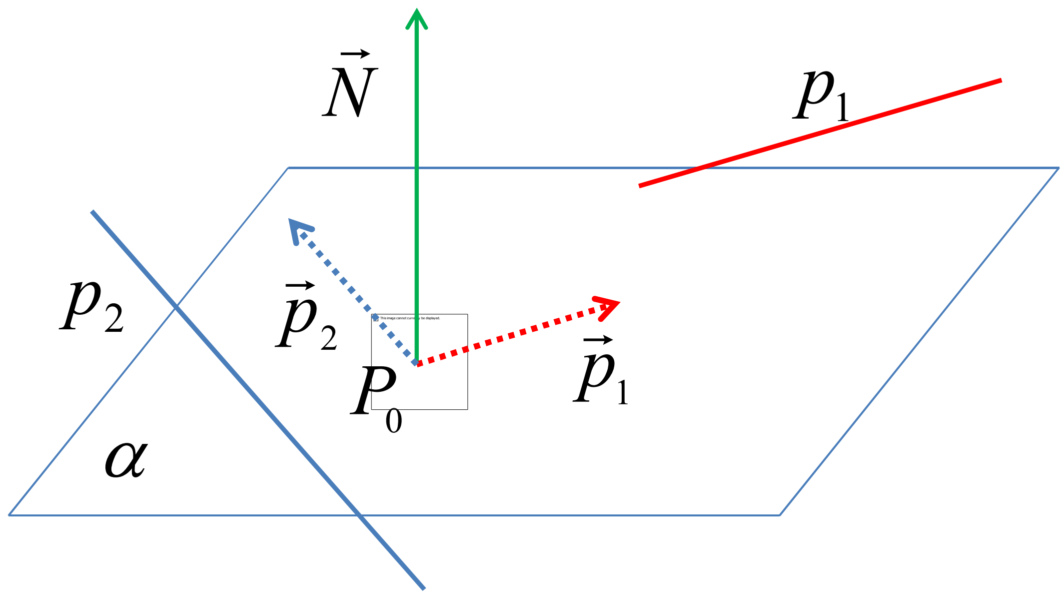
Napisati jednačinu ravni α
koja sadrži tačku

$P_0(2,3,0)$

i paralelna je pravama

$$p_1 : \frac{x+5}{1} = -\frac{y-5}{3} = -\frac{z-1}{2}$$ i

$$p_2 : x = y = z$$



$$\vec{N} = \vec{p}_1 \times \vec{p}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = -\vec{i} - 3\vec{j} + 4\vec{k}$$

$\alpha :$ $P_0(2,3,0)$ $\vec{N} = (-1,-3,4)$

$$-(x-2) - 3(y-3) + 4(z-0) = 0$$

$$-x - 3y + 4z + 11 = 0$$

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Napisati jednačinu ravni α

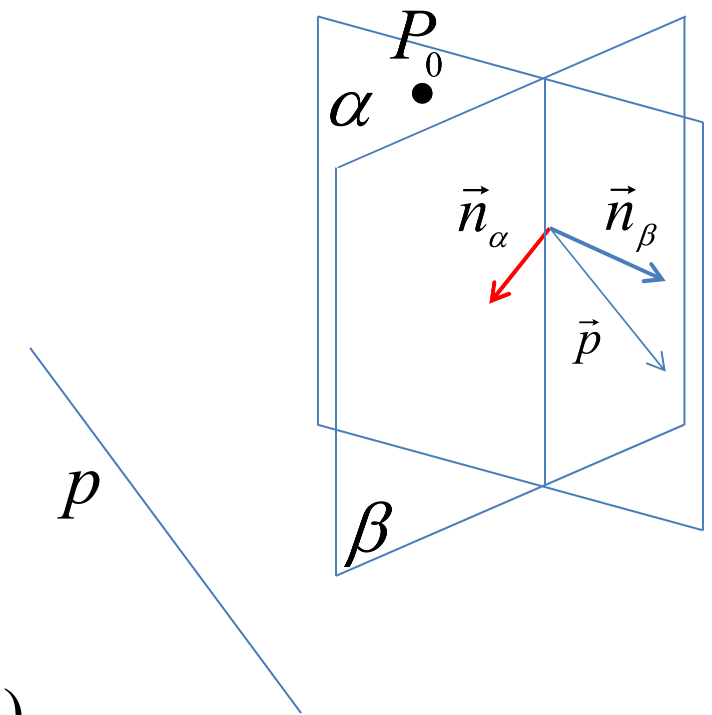
koja sadrži tačku $P_0(0,2,3)$,

normalna je na ravan

$\beta: 2x + y - 2z + 2 = 0$

i paralelna pravoj

$$p: \frac{x+2}{1} = \frac{y}{-3} = \frac{z}{1}$$



$\vec{a} = \vec{n}_\beta = (2,1,-2)$ $\vec{b} = \vec{p} = (1,-3,1)$

$$\vec{N} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & -3 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -5\vec{i} - 4\vec{j} - 7\vec{k}$$

$\alpha: \quad P_0(0,2,3) \quad \vec{N}_\alpha = (5,4,7)$

$$5(x-0) + 4(y-2) + 7(z-3) = 0$$

$$5x + 4y + 7z - 29 = 0$$